A FIRST COURSE IN DIFFERENTIAL EQUATIONS

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PREFACE

This course is intended for students who have taken a first course in calculus and presupposes some familiarity with the processes of algebra and analytic geometry. It aims to give a concise and clear account of the most useful methods of solution of differential equations and to provide ample material for practice in these methods. At the same time the student is kept in close touch with those applications of the subject to which it owes much of its development. In these applications the problems chosen have been confined to those in which a student with an elementary knowledge of mechanics can follow the formulation of the equations.

In the brief treatment of partial differential equations the topics have been selected with the object of indicating the possibilities of the subject and stimulating interest in its further study.

I desire to express my indebtedness to my colleagues, Dean J. Matheson and Professor C. F. Gummer, from whose suggestions I have profited, and to Mr. H. S. Pollock, M.Sc., who drew the figures.

N. M.

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CONTENTS

CHAPTER I

DEFINITIONS. FORMATION AND GEOMETRICAL MEANING OF DIFFERENTIAL EQUATIONS

1. Introduction and definitions. Equations involving two variables x and y have been studied in analytic geometry where they are seen to represent plane curves. It may be possible by solution of such an equation, $f(x, y) = 0$, to express y in terms of x by means of elementary functions.† Whether or not this is possible, the given equation, in case $f(x, y)$ is subject to certain very general conditions, determines y as a function of x, which from the nature of the relation is called an implicit function. In the study of this function the locus represented by the equation $f(x, y) = 0$ is of great use, as from it many properties of the function may be read off.

Equations frequently arise in which the relation among the variables (two or more in number) involves derivatives or differentials of these variables. These are *differential equations*. An *ordinary* differential equation is one in which only one variable is independent so that the derivatives which enter are all total. A *partial* differential equation is one in which two or more variables are independent so that the derivatives which enter are all partial.

Both ordinary and partial differential equations arise in many problems in the study of natural phenomena. In fact, in the majority of cases in which a scientific problem admits a mathematical statement, that statement is a differential equation. Many examples will appear in succeeding chapters.

The elementary study of differential equations is concerned with solving the simpler types, that is, with finding either in explicit or implicit form the functions which are defined by the equations. A solution is then any relation among the variables free from derivatives which, together with the derivatives obtained from it, reduces the equation to an identity. The term

† The so-called *elementary functions* include rational functions, irrational functions involving radicals, logarithms, exponential functions, and trigonometric functions direct and inverse.

solution is also applied to the locus (curve or surface) which this relation represents

The types of differential equations for which solutions can be found in closed form in terms of the elementary functions are comparatively few in number This raises the larger question as to the existence of functions satisfying differential equations of certain forms and the properties of such functions when they are not elementary The treatment of existence theorems is beyond the scope of this book Our point of view is the more elementary one with which we approach the study of algebraic equations, viz assume in the first place that the equation has a solution and find a relation among the variables which satisfies it, the fact that a differential equation is satisfied of course proves the existence of a solution for that individual case

The *order* of a differential equation is the order of the highest derivative which it contains

The *degree* of a differential equation of the nth order is the degree to which the nth derivative is raised when the equation is rational and integral in the derivatives

For the present we shall be concerned only with ordinary differential equations

An equation of the first order and first degree may be written in either of the forms

$$\frac{dy}{dx} = f(x, y)$$

or

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

EXAMPLE 1 The equation $\frac{dy}{dx} = -\frac{y}{x}$ or $x\,dy + y\,dx = 0$, of the first order and first degree is satisfied by $y = 1/x$, also by $y = c/x$ or $xy = c$ for any constant value of c

EXAMPLE 2 $\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0$ of the first order and second degree has the solutions $y = 2x$ and $y = 3x$ also the more general solutions $y = 2x + c$ and $y = 3x + c$ where c is any constant

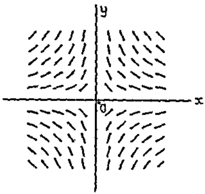
EXAMPLE 3 $\frac{d^2y}{dx^2} - y = 0$ of the second order and first degree

has the solutions $y = e^x$ and $y = e^{-x}$ and the more general solution $y = Ae^x + Be^{-x}$ where A and B are arbitrary constants. This may be verified by direct substitution.

2. Geometrical meaning of differential equations of the first order. When x and y are connected by a relation $f(x,y) = 0$, the derivative dy/dx gives at any point (x,y) the slope of the curve $f = 0$. Thus in order that a locus satisfy the equation $dy/dx = 1$ it must be subject to the single condition that at every point its slope is 1. It is obvious that all straight lines of slope 1 are solutions of this equation and that no other locus satisfies the equation at all its points. Again, the equation $\frac{dy}{dx} = \frac{y}{x}$ states that at any point the slope of the locus is the slope of a line joining that point to the origin. The solution then clearly consists of the pencil of all straight lines through the origin. Also a locus which at any point satisfies the equation $\frac{dy}{dx} = -\frac{x}{y}$ must cross at right angles the locus which belongs to the previous equation, from which it is evident geometrically that this last equation represents a set of concentric circles about the origin.

Consider now the general equation of the first order and first degree $dy/dx = f(x,y)$. If values be assigned arbitrarily to x and y a unique value will be determined by the equation for dy/dx. There is thus determined in the Cartesian plane a point (x,y) and a definite slope at that point. This combination of point and slope is called a line element. It may be represented on a figure by marking a point and drawing through it a short line segment with the proper slope. By drawing a sufficient number of these line elements in the plane we have a geometrical clue to the solution of the equation. Thus for the equation $x\frac{dy}{dx} + y = 0$ we obtain the adjoined figure. This figure strongly suggests the existence of a set of curves such that every point on each curve and the tangent to the curve at this point constitute a line element satisfying the differential equation. The Cartesian

equation of these curves is seen to be $xy = c$ (ex 1 art 1) where every constant value of c gives one curve of the set



A configuration of line elements in a plane is realized by the familiar experiment of sprinkling iron filings on a sheet of paper which is then held over a magnet The iron filings become magnetized and arrange themselves so that each one is a line element of the curve of force which passes through it

For a differential equation of the first order and higher degree there would correspond to each point two or more line elements and we infer that there would pass through every point two or more curves satisfying the equation

These geometrical considerations furnish strong evidence to the effect that a differential equation of the first order cannot represent only one curve but does in general represent a family of curves

3 The derivation of first order differential equations Further evidence of the fact just stated is furnished on the analytical side by the manner in which differential equations may be derived Suppose we have given an equation $F(x\ y\ c) = 0$ where c is an arbitrary constant or parameter that is for a definite value of c the equation represents a particular curve and different values assigned to c give different curves The

curves of such a set constitute a *one-parameter family*. By differentiation we obtain $\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx} = 0$. Between these two equations it is in general possible to eliminate the parameter c, the result of this elimination being a differential equation of the first order. It follows that every differential equation which may be obtained in this way (which certainly includes all first-order equations which have solutions in closed form in terms of elementary functions) represents a one-parameter family of curves.

For a first-order differential equation a solution which contains an arbitrary constant is called the *general solution* or *primitive*. A *particular solution* is any solution obtained from the general solution by giving to the arbitrary constant a particular value. It will later appear that there exist equations which have solutions that are not contained in the general solution. Such solutions are called *singular*.

Example 1. The equation $x^2+y^2-cx = 0$ with c arbitrary represents the one-parameter family of circles which touch the y-axis at the origin. By differentiation we get $2x+2y\frac{dy}{dx}-c = 0$ and, by eliminating c between the two equations,

$$x^2+y^2-x\left(2x+2y\frac{dy}{dx}\right) = 0$$

or

$$x^2-y^2+2xy\frac{dy}{dx} = 0,$$

which is the differential equation of the family.

Example 2. Find the differential equation for each of the following one-parameter families of curves:

(i) the family of straight lines of slope m,

(ii) the family of concurrent lines through the point $(0, b)$,

(iii) the family of equal circles of radius r with centres on the axis of x,

(iv) the family of equal circles of radius r with centres on the line $y = x$.

4. Differential equations of higher orders. A differential equation of the second order may be symbolized by $F(x,y,dy/dx,d^2y/dx^2)=0$ We shall take the equation to be of the first degree and leave to the student the variation in the argument for equations of higher degrees If values be arbitrarily assigned to x,y and dy/dx, the equation will determine a unique value for d^2y/dx^2 This value of d^2y/dx^2, together with the value assigned to dy/dx, fixes the value of $\dfrac{d^2y}{dx^2}\Big/\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$, which gives the curvature of the curve described by the point (x,y) The consequence is that at an arbitrary point and with an arbitrary slope at that point there exists a curve having a definite curvature If the function F is continuous in its four arguments, then a slight change in the position of the point (x,y) and in the slope dy/dx will produce only a slight change in the curvature Geometrically it is plausible therefore that the equation represents a family of curves of which an infinite number pass through every point All these curves will in general cut a given curve, say the x axis, and so the curves satisfying the equation are as numerous as all the directions which can be associated with all the points on a line To include all the solutions in a single primitive equation would necessitate two constants and the curves constitute a two parameter family

Conversely, it is clear that from a primitive equation containing two arbitrary constants a differential equation of the second order may be derived Thus from $f(x,y,a,b)=0$, in which a and b are constants, we obtain by two differentiations

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0,\qquad \frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0,$$

and from the three equations the two constants may be eliminated

From the results obtained in the cases of first and second order equations the corresponding facts regarding nth-order equations may be inferred by analogy and will be merely stated (1) From a primitive equation containing n arbitrary constants an equivalent differential equation of the nth order may be

derived. (2) The general solution of a differential equation of the nth order involves n arbitrary constants, or the equation represents an n-parameter family of curves.

EXAMPLE 1. The equation $(x-a)^2+(y-b)^2 = r^2$ with a and b arbitrary represents the family of all circles of radius r. To obtain the differential equation of this family we have

$$x-a+(y-b)\frac{dy}{dx} = 0$$

and
$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0,$$

from which $y-b = -\dfrac{1+(dy/dx)^2}{d^2y/dx^2}$, $x-a = \dfrac{1+(dy/dx)^2}{d^2y/dx^2}\dfrac{dy}{dx}$. Substitution of these values in the original equation gives

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}\Big/\frac{d^2y}{dx^2} = r;$$

which is seen to be a statement of the fact that at every point on every curve which it represents the curvature is constant and equal to $1/r$.

EXAMPLE 2. Find the differential equation corresponding to the equation of ex. 1 in which a and r are arbitrary constants and b is fixed.

EXAMPLE 3. Find the differential equation of all circles in the plane.

5. Boundary conditions. It has been seen that a differential equation has not a unique solution, that, in fact, the general solution contains a number of arbitrary constants equal to the order of the equation. When a differential equation is used in the solution of a problem which has a definite answer, the solution must satisfy one or more conditions besides the differential equation. These are called boundary conditions. Thus a first-order equation has in general a solution which satisfies the condition $y = \alpha$ when $x = x_0$; a second-order equation has a solution which satisfies the two conditions $y = \alpha$ and $dy/dx = \alpha_1$, when $x = x_0$, etc., the *boundary values* α, α_1, etc., being fixed constants.

EXAMPLE The equation $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = r\frac{d^2y}{dx^2}$ has been seen to have the general solution $(x-a)^2+(y-b)^2 = r^2$ Let us impose the two boundary conditions $y = 0$ when $x = 0$ and $dy/dx = 0$ when $x = 0$ Then $a^2+b^2 = r^2$ Also since

$$x-a+(y-b)dy/dx = 0$$

it follows that $a = 0$ from which $b = \pm r$ Hence the solution of the equation subject to the two boundary conditions consists of the two circles $x^2+y^2\pm 2ry = 0$ Further illustrations of boundary conditions will occur in the chapters on applications of differential equations

EXAMPLES ON CHAPTER I

Find the differential equations for the following families of curves

1 All straight lines in the plane

2 All straight lines tangent to the circle $x^2+y^2 = 1$

3 The family of spirals $r = ce^{\theta}$

4 The family of parabolas each with vertex at the origin and axis along the x-axis.

5 All circles in the first and third quadrants touching the coordinate axes

6 The family of lines each of which makes intercepts on the coordinate axes whose sum is constant

7 The family of lines each of which makes with the coordinate axes a triangle of constant area

8 A family of equal parabolas with axes along the x-axis.

9 All ellipses of constant area with axes along the coordinate axes

10 All circles coaxal with $x^2+y^2 = 1$ and $x^2+y^2 = 5x$

11 All conics confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

12 All circles of unit radius passing through the origin.

13 All circles with centres on the line $y = 2x$

14 All ellipses and hyperbolas with axes along the coordinate axes.

CHAPTER II

EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

THE general differential equation of the first order and first degree is $M\,dx + N\,dy = 0$ where M and N are functions of x and y. Some special cases will now be considered in which the general solution is readily found. The fact that these cases make up a comparatively small portion of all possible equations of this form is due to the nature of the subject. The functions defined by differential equations are in many cases not elementary functions and more advanced methods must be used in studying them.

6. Exact equations. If u is a function of x and y possessing first partial derivatives, then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ is called an exact differential. Thus, in order that $M\,dx + N\,dy$ be an exact differential it is necessary and sufficient that a function u exist such that $\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \equiv M\,dx + N\,dy$ or, in other words, such that $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$. It follows that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$ provided that this last derivative exists and is continuous. The condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is therefore necessary in order that the differential be exact. That this condition is also sufficient is not so obvious. The following proof makes use of a definite integral with a variable upper limit. Assuming that the condition holds, let us form the function $u = \int_{x_0}^{x} M(x, y)\,dx + \phi(y)$, where x_0 is any constant, y is independent of x in the integrand, and $\phi(y)$ is a differentiable function of y whose value will be assigned later. Now

$$\frac{\partial u}{\partial x} = M(x, y),$$

$$\frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial M}{\partial y}\, dx + \phi'(y) = \int_{x_0}^{x} \frac{\partial N}{\partial x}\, dx + \phi'(y)$$

$$= N(x\ y) - N(x_0\ y) + \phi'(y)$$

$\partial u/\partial y$ will be equal to N provided

$$\phi'(y) = N(x_0\ y) \quad \text{or} \quad \phi(y) = \int_{y_0}^{y} N(x_0\ y)\, dy,$$

where y_0 is some constant Hence $M\,dx + N\,dy$ is the exact differential of

$$u = \int_{x_0}^{x} M(x\ y)\, dx + \int_{y_0}^{y} N(x_0\ y)\, dy$$

If $M\,dx + N\,dy$ is an exact differential, the equation $M\,dx + N\,dy = 0$ is called an exact differential equation The solution of such an equation may usually be written down by inspection

EXAMPLE 1 $(3x^2 - 2y^2)\,dx - (4xy + 2y)\,dy = 0.$

This equation is found to be exact Inspection of the first term shows that the function whose differential is the left member of the equation must contain the terms $x^3 - 2xy^2$ and can have no other term containing x The only term in the equation unaccounted for is therefore $-2y\,dy$ which is the differential of $-y^2$ The solution of the equation is therefore $x^3 - 2xy^2 - y^2 = c$

EXAMPLE 2 $(2x^3 - 3x^2y + 4xy^3)\,dx + (y + 6x^2y^2 - x^3)\,dy = 0$

EXAMPLE 3 $(e^x + e^y + y\cos x)\,dx + (\sin x - \sin y + xe^y)\,dy = 0$

7 Separation of variables A simple case of exact differential equations is $\phi(x)\,dx + \psi(y)\,dy = 0$ which has the solution $\int \phi(x)\,dx + \int \psi(y)\,dy = c$ It follows that any process by which an equation may be thrown into this form leads to its solution The process is called separation of the variables

EXAMPLE 1 $\tan y \sec^2 x\, dx - \tan x\, dy = 0$

Here the variables are separated by dividing by $\tan x \tan y$ which gives

$$\frac{\sec^2 x}{\tan x}\, dx - \cot y\, dy = 0,$$

whence $\log\tan x - \log\sin y = c$, or $\tan x = C\sin y$ where $C = e^c$.

EXAMPLE 2. $\sqrt{(1-y^2)}\,dx + \sqrt{(1-x^2)}\,dy = 0$.

EXAMPLE 3. $(1+x)(1+y^2)\,dx + 2xy\,dy = 0$.

EXAMPLE 4. $(y^2+2y-3)\,dx + 2(x^2-4)(y+2)\,dy = 0$.

8. Homogeneous equations. This name is given to equations which have or may be made to take the form $dy/dx = F(y/x)$. This form is assumed by the equation $M\,dx + N\,dy = 0$ whenever M and N are homogeneous of the same degree since then M/N becomes a function of the single variable y/x. This suggests the substitution $y/x = v$ or $y = vx$, from which $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ and the equation becomes $v + x\dfrac{dv}{dx} = F(v)$, or $\dfrac{dv}{F(v)-v} = \dfrac{dx}{x}$, in which the variables are separate.

EXAMPLE 1. $(x+y)^2\,dx - 2x^2\,dy = 0$.

Set $y = vx$, $dy = v\,dx + x\,dv$. This gives

$$(1+v)^2\,dx - 2(v\,dx + x\,dv) = 0$$

or

$$(1+v^2)\,dx - 2x\,dv = 0.$$

Hence

$$\frac{dx}{x} - \frac{2dv}{1+v^2} = 0$$

and

$$\log x - 2\tan^{-1}v = c$$

or

$$\log x - 2\tan^{-1}\frac{y}{x} = c.$$

EXAMPLE 2. $(x+y)\,dx - (x-y)\,dy = 0$.

EXAMPLE 3. $(x^2+y^2)\,dx + xy\,dy = 0$.

EXAMPLE 4. $(x^3-2y^3)\,dx + 3xy^2\,dy = 0$.

9. Equations reducible to the homogeneous form. The equation $\dfrac{dy}{dx} = F\left(\dfrac{ax+by}{a_1x+b_1y}\right)$ is homogeneous. The more general equation $\dfrac{dy}{dx} = F\left(\dfrac{ax+by+c}{a_1x+b_1y+c_1}\right)$ reduces to this form when the origin is translated to the point of intersection of the lines $ax+by+c = 0$ and $a_1x+b_1y+c_1 = 0$. If, however, these lines

are parallel, that is, if $a_1x+b_1y = k(ax+by)$, then the substitution $ax+by = z$ from which $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx}-a\right)$ transforms the equation into $\frac{dz}{dx}-a = bF\left(\frac{z+c}{kz+c_1}\right)$ in which the variables are separable

In particular the equation

$$(ax+by+c)\,dx+(a_1x+b_1y+c_1)\,dy = 0$$

may always be solved by the methods of this section

EXAMPLE 1 $(x+y-1)^2\,dx-2(x-2)^2\,dy = 0$

The lines $x+y-1 = 0$ and $x-2 = 0$ intersect at $(2, -1)$ Hence set $x = X+2$, $y = Y-1$ This gives

$$(X+Y)^2\,dX-2X^2\,dY = 0$$

which is ex 1, art 8

EXAMPLE 2 $(x-y+1)\,dx-(2x-2y-3)\,dy = 0$

Set $x-y = z$, $dy = dx-dz$ Then

$$(z+1)\,dx-(2z-3)(dx-dz) = 0$$

or $(4-z)\,dx+(2z-3)\,dz = 0$, which is easily integrated after dividing by $4-z$

EXAMPLE 3 $(6x+4y-8)\,dx+(x+y-1)\,dy = 0$

EXAMPLE 4 $(3x-y+4)\,dx+(2y-6x+3)\,dy = 0$

EXAMPLE 5 $(x+y+2)^2\,dx+(x+y)^2\,dy = 0$

10. Integrating factors. If the equation

$$\mu(x,y)(M\,dx+N\,dy) = 0$$

is exact, the function μ is called an integrating factor of the equation $M\,dx+N\,dy = 0$, which is now supposed not exact Thus, for example, any function the multiplication by which separates the variables in an equation is an integrating factor of that equation In other equations an integrating factor may sometimes be found by inspection The equation $y\,dx-x\,dy = 0$ is rendered exact by multiplying it by $1/xy$, which separates the variables, or by $1/y^2$ since $\frac{y\,dx-x\,dy}{y^2} = d\frac{x}{y}$, or by $1/x^2$ since

$\frac{y\,dx-x\,dy}{x^2}=-d\frac{y}{x}$. In fact $\frac{1}{y^2}f\left(\frac{x}{y}\right)$, where f is any integrable function, is also an integrating factor since it transforms the equation into $f(x/y)\,d(x/y)=0$. This illustrates the general fact that if an equation has an integrating factor it has an unlimited number of such factors. For certain special types of equations rules for finding integrating factors may be given. In these cases, however, it will generally be found that the factors can be discovered without using the rules. Except in the case of the linear equation (art. 11) the finding of integrating factors will be left to the student's ingenuity with the suggestions given in the following examples.

EXAMPLE 1. $(x^3+y)\,dx-x\,dy=0$.

The term $x^3\,dx$ is an exact differential and will remain so on multiplication by a function of x. Hence a function of x which makes $y\,dx-x\,dy$ exact is an integrating factor of the equation. Such a function is obviously $1/x^2$, which gives the equation $x\,dx+\frac{y\,dx-x\,dy}{x^2}=0$. The solution is $\frac{x^2}{2}-\frac{y}{x}=\frac{c}{2}$ or $x^3-2y=cx$.

EXAMPLE 2. $(x^2+y^2+x)\,dx-(2x^2+2y^2-y)\,dy=0$.

The terms may be grouped thus,

$$(x^2+y^2)(dx-2dy)+x\,dx+y\,dy=0,$$

from which it appears that $1/(x^2+y^2)$ is an integrating factor giving the solution $x-2y+\frac{1}{2}\log(x^2+y^2)=c$.

EXAMPLE 3. $my\,dx+nx\,dy=0$.

The variables here are separable. It is useful, however, to find a general integrating factor, as follows. Since

$$d(x^{km}y^{kn})=kx^{km-1}y^{kn-1}(my\,dx+nx\,dy),$$

it follows that $x^{km-1}y^{kn-1}$ is an integrating factor of the given equation for any value of k. The fact that this constant k is at our disposal enables us to find an integrating factor for the equation

$$x^{\alpha}y^{\beta}(my\,dx+nx\,dy)+x^{\alpha_1}y^{\beta_1}(m_1y\,dx+n_1x\,dy)=0.$$

For integrating factors for the first and second groups of terms are respectively $x^{km-1-\alpha}y^{kn-1-\beta}$ and $x^{k_1m_1-1-\alpha_1}y^{k_1n_1-1-\beta_1}$. The

same integrating factor is obtained for both groups of terms if k and k_1 are chosen to satisfy the equations

$$km-1-\alpha = k_1 m_1 - 1 - \alpha_1$$
$$kn-1-\beta = k_1 n_1 - 1 - \beta_1$$

EXAMPLE 4 $x^2(y\,dx+2x\,dy)-y^2(y\,dx+4x\,dy)=0$

EXAMPLE 5 $y\,dx+x(xy^2+1)\,dy=0$

EXAMPLE 6 $(x^2y^2-y)\,dx+(2x^3y+x)\,dy=0$

EXAMPLE 7 $(7y+3xy^2)\,dx-(8x+2x^2y)\,dy=0$

11 Linear equations A linear differential equation of any order is one which is linear in the dependent variable and its derivatives The linear equation of the first order may be written $\frac{dy}{dx}+Py=Q$ where P and Q are functions of x Consider first the special case in which Q is zero $\frac{dy}{dx}+Py=0$ By separating the variables we obtain the solution $\log y+\int P\,dx=c$ or $ye^{\int P\,dx}=C$ where $C=e^c$ Differentiation of this result gives $e^{\int P\,dx}\left(\frac{dy}{dx}+Py\right)=0$ which shows that $e^{\int P\,dx}$ is an integrating factor of the equation just solved But since it involves only x $e^{\int P\,dx}$ is also an integrating factor of $\frac{dy}{dx}+Py=Q$ Making use of this factor we find for the solution of the general equation $ye^{\int P\,dx}=\int Qe^{\int P\,dx}\,dx+c$ †

EXAMPLE 1 $\frac{dy}{dx}+y\cot x=\sin x$

Here $\int P\,dx=\int \cot x\,dx=\log\sin x$

and $e^{\log\sin x}=\sin x$

Hence $y\sin x=\int \sin^2 x\,dx=\frac{1}{2}(x-\sin x\cos x)+c$

† Since any value of the indefinite integral $\int P\,dx$ makes $e^{\int P\,dx}$ an integrating factor there is no need to consider a constant of integration. Moreover if in this formula $\int P\,dx$ be replaced by $\int P\,dx+k$ it is readily seen that the solution is no more and no less general

EXAMPLE 2. $\dfrac{dy}{dx}+\dfrac{y}{x}=x^3$.

EXAMPLE 3. $(1+x^2)\dfrac{dy}{dx}+xy=\sqrt{(1+x^2)}$.

EXAMPLE 4. $\dfrac{dy}{dx}+xy=x$.

12. Equations reducible to the linear form. If in the equation $\dfrac{dy}{dx}+Py=Q$, y be replaced by a function of y, the solution of art. 11 will give the value of this function of y. Thus $2y\dfrac{dy}{dx}+Py^2=Q$ has the solution $y^2e^{\int P\,dx}=\int Qe^{\int P\,dx}\,dx+c$. This is a particular case of a more general form known as Bernoulli's equation, viz. $\dfrac{dy}{dx}+Py=Qy^n$, which takes the form $y^{-n}\dfrac{dy}{dx}+Py^{1-n}=Q$ or $\dfrac{dy^{1-n}}{dx}+(1-n)Py^{1-n}=(1-n)Q$ which is linear in y^{1-n} as dependent variable.

In the equation $M\,dx+N\,dy=0$ either x or y may be regarded as the independent variable. It is sufficient for the purpose of solution if the equation takes or may be made to take the linear form either when x or when y is regarded as the independent variable.

EXAMPLE 1. $\dfrac{dy}{dx}+\dfrac{y}{x}=x^3y^4$.

Dividing by y^4 gives $y^{-4}\dfrac{dy}{dx}+\dfrac{1}{x}y^{-3}=x^3$ or

$$\frac{dy^{-3}}{dx}-\frac{3}{x}y^{-3}=-3x^3.$$

Now $\int -\dfrac{3}{x}dx=-3\log x=\log x^{-3}$ and $e^{\log x^{-3}}=x^{-3}$.

Hence

$$\begin{aligned}x^{-3}y^{-3}&=\int -3x^3x^{-3}\,dx+c\\&=-3x+c.\end{aligned}$$

EXAMPLE 2 $\quad (xy^2-y^2-y^5)\dfrac{dy}{dx}+1+y^3=0$

If y be taken as independent variable, this equation is

$$(1+y^3)\frac{dx}{dy}+y^2x=y^2+y^5 \quad\text{or}\quad \frac{dx}{dy}+\frac{y^2}{1+y^3}x=y^2$$

Since $\displaystyle\int\frac{y^2}{1+y^3}\,dy=\log(1+y^3)^{\frac{1}{3}}$ the integrating factor is $(1+y^3)^{\frac{1}{3}}$ and the solution is therefore

$$x(1+y^3)^{\frac{1}{3}}=\int(1+y^3)^{\frac{1}{3}}y^2\,dy+c$$
$$=\tfrac{1}{4}(1+y^3)^{\frac{4}{3}}+c$$

EXAMPLE 3 $\quad \dfrac{dy}{dx}-x^2y=y^4e^{-\frac{1}{3}x^3}$

EXAMPLE 4 $\quad 2\dfrac{dx}{dy}+\dfrac{x}{\sqrt{(1+y^2)}}=\dfrac{\sqrt{(1+y^2)}-y}{x}$

13 Change of variables The student will recall from his study of the integral calculus that one of the most effective means of evaluating indefinite integrals is to make a suitable change of the variable of integration This principle has even wider application in the solution of differential equations The solution of the homogeneous equation (art 8) was effected by the change of the dependent variable from y to v where $y=vx$, while the method of reducing Bernoulli's equation (art 12) to the linear form amounts to the substitution $y^{1-n}=v$ It may happen that the solution of a differential equation is facilitated by a change of either one or of both variables If the original variables are x and y and one of them, say x, is replaced by u, then u will be some function of x and y or of x alone When the student has ascertained that a differential equation does not conform to any of the simple types his next effort should be to simplify its form by a change of one or of both variables For this no precise rules can be given Close attention to the manner in which the variables occur in the equation together with a little experience will often suggest a useful substitution

EXAMPLE 1. $(1+xy)\,dx-2x^2(1-2xy)\,dy = 0.$

If this equation be divided by x^3 it takes the form

$$\left(\frac{1}{x}+y\right)\frac{dx}{x^2}-2\left(\frac{1}{x}-2y\right)dy = 0$$

which suggests the substitution $u = 1/x$. Then

$$(u+y)\,du+2(u-2y)\,dy = 0$$

which is homogeneous.

EXAMPLE 2. $\dfrac{x\,dx+y\,dy}{x\,dy-y\,dx} = \sqrt{\left(\dfrac{a^2-x^2-y^2}{x^2+y^2}\right)}.$ Imp

Set $x = r\cos\theta$, $y = r\sin\theta$. Then

$$x^2+y^2 = r^2 \quad\text{and}\quad x\,dx+y\,dy = r\,dr.$$

Also

$$x\,dy-y\,dx = r\cos\theta(r\cos\theta\,d\theta+\sin\theta\,dr)-$$
$$-r\sin\theta(-r\sin\theta\,d\theta+\cos\theta\,dr) = r^2\,d\theta.$$

The equation now becomes

$$\frac{r\,dr}{r^2\,d\theta} = \frac{\sqrt{(a^2-r^2)}}{r}$$

or

$$\frac{dr}{\sqrt{(a^2-r^2)}} = d\theta$$

from which $\sin^{-1}\dfrac{r}{a} = \theta+c$ or $r = a\sin(\theta+c)$.

EXAMPLES ON CHAPTER II

1. $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}.$
2. $(y^3+2y^2x+x^3)\,dx+x^2(y+3x)\,dy = 0.$
3. $x\dfrac{dy}{dx}-2x^2y = e^{x^2}.$
4. $(2xy+x^2)\,dy+(x^3-y^3)\,dx = 0.$
5. $\dfrac{dx}{d\theta}-x\tan\theta = x^4\sin\theta.$
6. $x(3xy^2+2y^3+3x)\,dx+y(2x^3+3x^2y-3y)\,dy = 0.$
7. $(x+a)^2\dfrac{dy}{dx}+3(x+a)y = b.$
8. $(2x-5y+2)\,dx+(10y-4x)\,dy = 0.$
9. $x\,dy+y\,dx = (x^2y^2+xy)\,dx.$
10. $r\sin\theta\,dr+(r^2+1)\cos\theta\,d\theta = 0.$

11 $(y\sec^2 x-\tan^2 x)\,dx+\tan x\,dy=0$

12 $ye^{-x/y}\,dx-(xe^{-x/y}+y^3)\,dy=0$

13 $\{2\tan s\sec^2 s-t\sin s-\cos(s+t)\}\,ds+\{\cos s-\cos(s+t)-e^t\}\,dt=0$

14 $y^2\,dx-(xy+1)\,dy=0$

15 $(3x^2y+2y^4)\,dx-(x^3-4xy^3)\,dy=0$

16 $(e^{3y}-3x)\,dy=dx$

17 $x\,dy-(y+x\tan y/x)\,dx=0$

18 $\dfrac{dy}{dx}=(a^2x+b^2y)^2$

19 $x\log x\dfrac{dy}{dx}+2y=4\sqrt{y}\log x$

20 $xy^2\,dx+(x^3+y^3+y^2)\,dy=0$

21 $\sqrt{(x^2+y^2)}(x\,dy-y\,dx)=a(x\,dx+y\,dy)$

22 $3y^2\dfrac{dy}{dx}=x^2(y^3+1)$

23 $(x+2y+1)\,dx+(2x+y-4)\,dy=0$

24 $x\dfrac{dy}{dx}+y\log y=xye^x$

25 $x^2y\dfrac{dy}{dx}+y^2=x^4$

26 $\dfrac{dr}{d\theta}=\dfrac{1}{2}\left(\dfrac{\tan\theta}{r}-r\sec\theta\right)$

27 $(2x-y)\,dx+(1+x)(y-x)\,dy=0$

28 $y^2(9y\,dx+12x\,dy)+x^3(12y\,dx+8x\,dy)=0$

29 $2(y+2)^2\,dx-(x+y-3)^2\,dy=0$

30 $(e^y\tan x+2y\tan^2 x)\,dy-e^y\sec^2 x\,dx=0$

CHAPTER III

EQUATIONS OF THE FIRST ORDER BUT NOT OF THE FIRST DEGREE

If a differential equation of the first order is not of the first degree this circumstance in general renders the equation more difficult of solution. Our effort will be directed towards making the solution depend upon that of one or more equations of the first order and first degree. Certain devices will now be given which in this way effect the solution of a limited number of types of equations. It will be convenient to denote the first derivative which occurs in the equation by p.

14. Equations solvable for p. If an equation can be solved for p in terms of functions of x and y the equation is replaced by one or more equations of the first degree. The equation $p^n+P_1p^{n-1}+P_2p^{n-2}+\dots+P_{n-1}p+P_n=0$, where the P's are functions of x and y, is of the first order and nth degree. It may be feasible to express this equation in the factored form $(p-Q_1)(p-Q_2)\dots(p-Q_n)=0$, where the Q's are functions of x and y. Then every solution of each of the equations $p-Q_i=0$ is a solution of the given equation and every solution of the given equation is a solution of one of these first-degree equations. If now it is possible to solve each of the equations $p-Q_i=0$, the solutions being $F_i(x,y,c)=0$ $(i=1,2,\dots,n)$, then the general solution of the given equation is

$$F_1(x,y,c)F_2(x,y,c)F_3(x,y,c)\dots F_n(x,y,c)=0.\dagger$$

Example 1. $p^2-(3x^2-2xy)p-6x^3y=0$.

This equation is equivalent to $(p-3x^2)(p+2xy)=0$. $\frac{dy}{dx}-3x^2=0$ has the solution $y-x^3-c=0$ and $\frac{dy}{dx}+2xy=0$ has the solution $y-ce^{-x^2}=0$. Hence the general solution of the given equation is $(y-x^3-c)(y-ce^{-x^2})=0$.

† Two particular solutions may be $F_1(x,y,c_1)=0$ and $F_2(x,y,c_2)=0$ with different values of c, but the totality of all particular solutions is evidently given by using the same arbitrary constant c in each factor.

EXAMPLE 2 $p^2 = 9x^2+9x^4$

This equation resolves into

$$\{p-3x\sqrt{(1+x^2)}\}\{p+3x\sqrt{(1+x^2)}\} = 0,$$

and gives the general solution

$$\{y+c-(1+x^2)^{\frac{3}{2}}\}\{y+c+(1+x^2)^{\frac{3}{2}}\} = 0$$

or

$$(y+c)^2 = (1+x^2)^3$$

EXAMPLE 3 $p^3-6p^2+8p = 0$

EXAMPLE 4 $p^2 = 1-y^2$

EXAMPLE 5 $p^2-(x+y)p+xy = 0$

EXAMPLE 6 $xyp^2-(x^2+y^2)p+xy = 0$

15 Equations solvable for y Consider an equation which may be thrown into the form

$$y = f(x, p) \tag{1}$$

In order to devise a method of solving it, let a solution be assumed in the form $y = \phi(x)$ If this be granted we get by substitution in (1)

$$\phi(x) = f(x, \phi'(x)), \tag{2}$$

whence the solution is $y = f(x, \phi'(x))$ It follows that the solution is obtained by the substitution of a suitable function $\phi'(x)$ for p in (1) It remains to determine this function $\phi'(x)$ From (2) it must satisfy the relation

$$\phi' = \frac{\partial f}{\partial x}+\frac{\partial f}{\partial \phi}\frac{d\phi'}{dx}, \tag{3}$$

and this equation, provided it can be solved, suffices to determine ϕ'

If in (3) $\phi'(x)$ be replaced by p, the equation is the result of differentiating (1) with regard to x, that is,

$$p = \frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{dp}{dx} \tag{4}$$

Hence the method may be condensed into the following rule Differentiate the equation with regard to x Solve the resulting equation for p in terms of x and substitute this value of p in the given equation If the solution of (4) is found in the implicit form

$$F(x, p, c) = 0, \tag{5}$$

the required solution is found by eliminating p between (1) and (5). If this elimination is not easily made it is preferable to regard the two equations as parametric equations of the solution, p acting in this case as a parameter.†

EXAMPLE 1. $y = x+p^2-2p$.

By differentiation with respect to x we get

$$p = 1+(2p-2)\frac{dp}{dx}$$

or
$$(p-1)\left(2\frac{dp}{dx}-1\right) = 0,$$

whence $p = 1$ or $2p = x+c$. $p = 1$ gives $y = x-1$ which is a solution but, since it contains no arbitrary constant, is not the general solution. $p = \frac{1}{2}(x+c)$ gives the general solution, viz. $y = x+\frac{1}{4}(x+c)^2-x-c$ or $(x+c)^2 = 4(y+c)$.

EXAMPLE 2. $2y+x^2 = 2p^2$.

By differentiation we get

$$2p+2x = 4p\frac{dp}{dx} \quad \text{or} \quad (p+x)\,dx = 2p\,dp.$$

The substitution $p = vx$ leads to the solution

$$(p-x)^2(2p+x) = c.$$

Now p may be eliminated between this and the given equation or the two equations together may be regarded as parametric equations of the general solution.

EXAMPLE 3. $2y+x^4 = xp$.

EXAMPLE 4. $y+xp = x^4p^2$.

16. Equations solvable for x. The method here is analogous to that used in the preceding case. If the equation may be made to take the form

$$x = f(y,p), \tag{1}$$

† The student should distinguish carefully the three meanings of p in this article. In (1) p stands for dy/dx; in (4) p is the dependent variable, i.e. a function of x which satisfies this differential equation; finally (5) and (1) constitute the general solution of (1) when p is a parameter in the sense of that term in the study of parametric equations of curves.

assume that $x = \psi(y)$ is a solution Then

$$\psi(y) = f\left(y \; \frac{1}{\psi(y)}\right) \tag{2}$$

and the solution is
$$x = f\left(y, \frac{1}{\psi(y)}\right) \tag{3}$$

Now replace the function $1/\psi(y)$ by $p(y)$ or p for short Then by differentiating (2) we get

$$\frac{1}{p(y)} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}, \tag{4}$$

which is a differential equation to determine $p(y)$ If this equation can be solved for $p(y)$, the solution of (1) is $x = f(y, p(y))$ It will now be observed that (4) is the equation resulting from differentiating (1) with regard to y and writing $dx/dy = 1/p$ Hence the rule if a differential equation is solved for x, differentiate with regard to y and solve the resulting differential equation for p, this solution, together with the given equation, form parametric equations of the general solution, from which the parameter p may be eliminated if this is feasible

EXAMPLE 1 $x = y+2p-p^2$

We have here a choice of differentiating with regard to either x or y Differentiating with regard to y we get

$$\frac{1}{p} = 1+2\frac{dp}{dy} - 2p\frac{dp}{dy},$$

or $\dfrac{1-p}{p} = 2(1-p)\dfrac{dp}{dy}$ Either $p = 1$ or $dy = 2p\,dp$

The latter equation gives $y+c = p^2$ Hence the general solution results by substituting $p = \pm\sqrt{(c+y)}$ in the given equation This gives

$$x = \pm 2\sqrt{(y+c)} - c \quad \text{or} \quad (x+c)^2 = 4(y+c)$$

EXAMPLE 2 $x = yp^2+p^3$

EXAMPLE 3 $p(x-y^3) = y$

17 Clairaut's equation Among the equations to which the method of art 15 applies, the equation $y = xp+f(p)$ known as

Clairaut's form, deserves special notice. Differentiation of the equation gives

$$p = x\frac{dp}{dx}+p+f'(p)\frac{dp}{dx},$$

or

$$\frac{dp}{dx}(x+f'(p)) = 0,$$

which is satisfied if either factor vanishes. $dp/dx = 0$ gives $p = c$, whence $y = cx+f(c)$ is the general solution. A solution may also be obtained by eliminating p between $x+f'(p) = 0$ and the given equation. The nature of this solution will be explained in the following chapter.

It is sometimes possible by a change of one variable or of both to transform an equation into Clairaut's form. The general solution is then at once written down.

EXAMPLE 1. $y = 2px+yp^2$.

This equation is not in Clairaut's form as it stands. Multiply it by y. Then $y^2 = 2ypx+y^2p^2$. Now set $y^2 = v$. Then $v = x\frac{dv}{dx}+\frac{1}{4}\left(\frac{dv}{dx}\right)^2$, whence the solution is $y^2 = cx+\frac{1}{4}c^2$, or $y^2 = 2Cx+C^2$ where $C = \frac{1}{2}c$.

CHAPTER IV

SINGULAR SOLUTIONS OF FIRST ORDER EQUATIONS

18 Illustration. Let us consider again the equation of ex 1, art 15,

$$y = x + p^2 - 2p \tag{1}$$

The general solution of this equation was found to be

$$(x+c)^2 = 4(y+c), \tag{2}$$

which denotes a family of equal parabolas with axes parallel to the y axis and vertices on the line $y = x$ The method of solution gave also

$$y = x - 1, \tag{3}$$

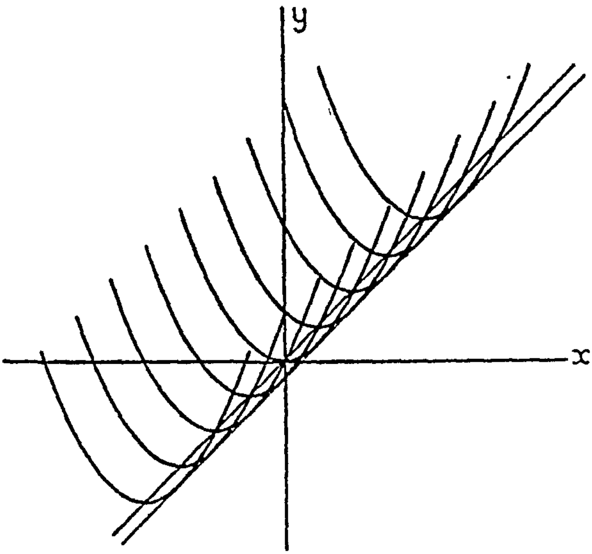
and we proceed to examine the relation of this locus to the equations (1) and (2) In the first place the line (3) is a solution of (1) since along it $p = 1$ and the substitution of $x - 1$ for y and 1 for p reduces (1) to an identity Secondly the line (3) touches each of the parabolas (2), for the substitution $x = y + 1$ in (2) gives $(y+1+c)^2 = 4(y+c)$ or $(y+c-1)^2 = 0$ and the two values of y are equal It is now clear why the line (3) is a solution of the differential equation although it is not one of the curves given by the general solution For every point (x, y) on it together with its slope $p = 1$ constitute a line element belonging to one of the parabolas (2) The line is an envelope of the family of curves which make up the general solution

A solution of a differential equation which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant is called a *singular solution* If the family of curves which constitute the general solution has an envelope, it is clear that the envelope is a singular solution It will now be assumed without further investigation that all singular solutions are of this nature

From the fact that the singular solution is a single curve as opposed to an infinite set of curves of the general solution it is not to be inferred that it is of little importance In many problems the greatest interest attaches to the singular solution and no equation which has a singular solution is considered

completely solved until all the solutions, singular as well as general, are found.

We turn now to a consideration of methods of determining the singular solution if one exists. For this purpose we may start either from the differential equation or from the general solution. The purpose of the following argument is to show what the facts are in the cases which are ordinarily met with. A complete theory of singular solutions presents difficulties



which can be appreciated only by one who has a grasp of the salient facts.†

19. The p-discriminant. Returning to the example of art. 18, consider a point P on the upper side of the line $y = x-1$. Through this point pass two parabolas having two different slopes or two different values of p. As the point P approaches the envelope the inclinations of the two tangents through P become more and more nearly equal. Finally, when P is on the envelope the two slopes coincide with that of the envelope.

† See E. L. Ince, *Ordinary Differential Equations*, London, 1927, p. 87, where detailed references to the literature are given.

The envelope is therefore distinguished by the fact that at each point on it two values of p are equal which at non specialized points are different The locus will therefore be determined from the differential equation by expressing the condition which must hold between x and y in order that the equation may give equal values for p In the example under discussion the condition is easily found by writing the equation as a quadratic in p, $p^2-2p+x-y=0$ The condition for equal roots of this quadratic is $1-x+y=0$ which is the singular solution already found

From the argument used in the case of the foregoing example we conclude that if the equation $f(x,y,p)=0$ has a singular solution that solution will be found by expressing the condition on x and y which must hold in order that this equation should have equal roots for p It does not follow conversely, however, that every locus which is determined in this way is a singular solution Other possibilities will be considered in art 21

For an equation quadratic in p, $Ap^2+Bp+C=0$ the condition necessary and sufficient for equal roots is $B^2-4AC=0$ Generally, for an algebraic equation of any degree, a function of the coefficients is defined whose vanishing determines that the equation has two or more equal roots This function is called the discriminant As applied to the equation in p, $f(x,y\ p)=0$, we shall call it the p discriminant and the equation obtained by equating it to zero the p-discriminant relation From the theory of equations we learn that in order that $f(x,y,p)=0$ have two or more equal roots in p, this root must also satisfy the equation $\partial f/\partial p=0$ whence the p-discriminant relation is obtained by eliminating p between these two equations †

In the solution of Clairaut's equation (art 17), $y=px+f(p)$ it appeared that a solution was obtained by eliminating p between the differential equation and $x+f'(p)=0$ The latter is now seen to be the result of differentiating the given equation

† If it should happen that the envelope is a line parallel to the y axis, then along this locus $dx/dy=1/p=0$ The p discriminant relation should give all the loci along which p or $1/p$ have equal finite values.

partially with regard to p. Hence according to our present argument the solution obtained is singular.

From the above discussion it follows that one method of searching for singular solutions is to find first the p-discriminant relation. This relation may resolve itself into a number of component equations. Each of these equations should then be tested to see whether it is a solution of the differential equation.

EXAMPLES. Find the p-discriminant relation for each of the following equations and test whether it is a solution.

1. $p^2+xp-y = 0.$
2. $yp^2-2yp+x = 0.$
3. $y^2p^2-2xyp = x^2-2y^2.$
4. $p^3-3px+3y = 0.$

20. The c-discriminant. The *singular solution of a differential equation may also be found from the general solution.* The problem is here that of determining the envelope of a one-parameter family of curves. This problem is discussed in books on calculus, to which the student should refer to refresh his memory. The family of curves $F(x, y, c) = 0$ with parameter c being given, the method consists in differentiating the equation with regard to c and eliminating c between the two equations $F(x, y, c) = 0$ and $\partial F/\partial c = 0$. If the family of curves has an envelope it will be found in this way, but again it does not follow that every locus which results from the elimination will be an envelope. Each locus found should be tested by substitution in the differential equation.

The method of finding the singular solution from the general solution is the same as from the differential equation. The process gives us the *c-discriminant* and the *c-discriminant relation.* The latter may be interpreted as the condition that the equation $F(x, y, c) = 0$ have equal roots in c. That this property pertains to the envelope is evident from the figure on p. 25. Through every point above the line $y = x-1$ there pass two curves given by two different values of c in the general solution. For a point on the envelope, however, the two curves through it coincide.

It follows that the singular solution may be thought of as

a locus characterized by equal values of p in the differential equation or by equal values of c in the general solution

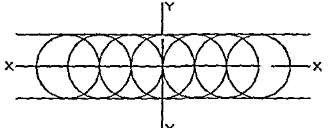
EXAMPLE 1 Find the singular solution of

$$(y-px)^2 = 1+p^2$$

either from the differential equation or the general solution

EXAMPLE 2 Solve examples 3 and 4 p 27 and find the singular solutions from the general solutions

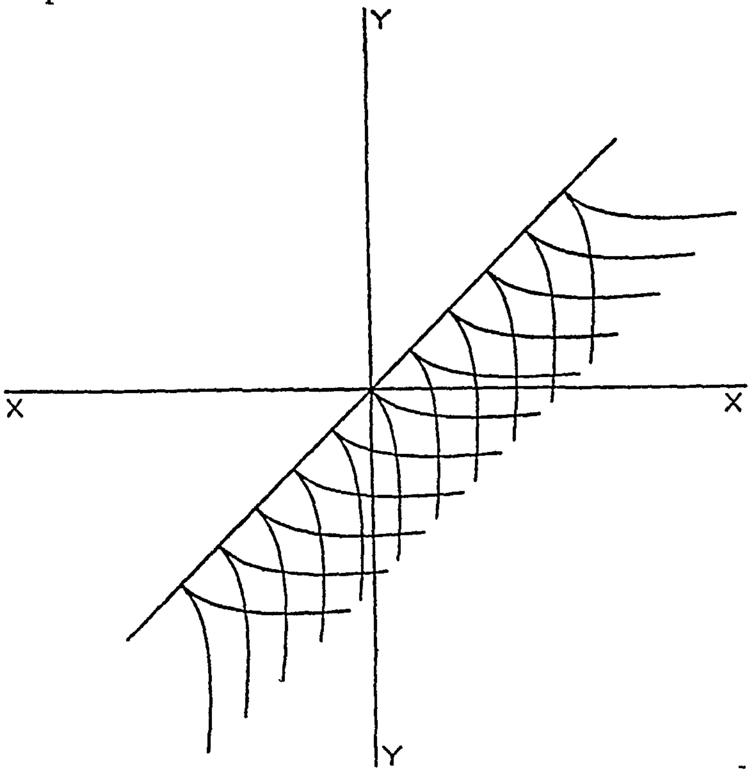
21 Tac cusp and node loci It has been pointed out that the arguments used in the last two articles do not ensure that



every locus found from the p and c-discriminant relations is a singular solution It is interesting to observe certain other types of loci which may occur

As a first example the equation $y^2(p^2+1) = a^2$ has the general solution $(x-c)^2+y^2 = a^2$ which denotes a set of circles of radius a with centres on the axis of x The p-discriminant relation is $y^2(y^2-a^2) = 0$ which is composed of three loci $y = 0$ $y = a$ and $y = -a$ The last two satisfy the differential equation and are envelopes of the set of circles $y = 0$ is not a solution of the differential equation Through any point on this locus pass two circles of the family whose tangents at that point coincide But the tangent to the circles is not tangent to the locus This explains why the locus occurs in the p-discriminant relation without being a solution of the equation Such a locus along which two or more curves of the family touch each other without touching the locus is called a *tac locus* Since curves which pass through any point on the tac locus are all distinct

it is not a locus along which values of c are equal and hence is not given by the c-discriminant. In the present example the c-discriminant relation is $y^2-a^2=0$ which gives only the envelope.



At each point on a tac locus two distinct curves of the family touch each other. A locus may exist at every point of which a curve of the family has contact with itself, which happens at a cusp on the curve. Consider the example

$$(1-p)^2(x-y) = (1+p)^2.$$

The substitutions $x+y = Y$, $x-y = X$ lead readily to the solution $4(x-y)^3 = 9(x+y+c)^2$. The p-discriminant is $x-y$ and the c-discriminant $(x-y)^3$. The line $x-y=0$, however, does not satisfy the differential equation. It is a locus of cusps of

the curves of the family Through any point below the line $y = x$ there pass two curves with two distinct values of p As the point approaches the envelope the curves through it approach coincidence and the two values of p become equal This example makes clear that a locus of cusps if one exists will occur in both the p and c discriminant relations but will not ordinarily be a singular solution In an exceptional case the curves at the cusps may be tangent to the cusp locus in which case this locus is also an envelope

In the study of singular points of curves a type of singularity other than a cusp is a node that is a point where a curve crosses itself If each curve of a one parameter family should have a node will the locus of nodes be a solution of the differential equation? Will its equation be contained in the p or c discriminant relations? The answers to these questions can be inferred from a study of the figure on p 31 in which the axis of x is a node locus Through each point near the x-axis pass two curves which approach coincidence as the point approaches the x axis Hence every point on the node locus is characterized by the fact that corresponding to it is only one curve of the family instead of two in other words two values of c are equal in the general solution The equation of the node locus will therefore occur in the c-discriminant relation. On the other hand the two values of p at a point on the node locus are distinct from each other as they are at other neighbouring points The p discriminant relation accordingly will not contain the equation of the node locus and the latter is not a solution of the differential equation An exception to this statement arises in the rare case in which the node locus is tangent at each node to one branch of the curve passing through it

22 Summary and examples Our result is that in the cases where such loci exist the p discriminant relation contains the equations of

the envelope
the tac locus
the cusp locus

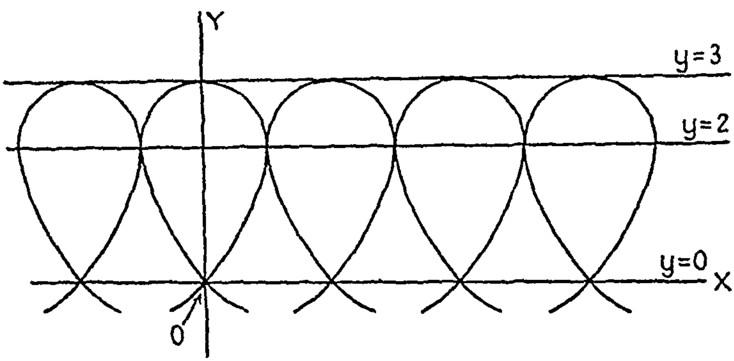
and the c-discriminant relation contains the equations of

the envelope,

the cusp locus,

the node locus.

Two further remarks may be added. A differential equation of the first degree in p containing only single-valued functions of x and y denotes a family of curves of which only one passes through each point of the plane. Such a family can have no envelope nor any other locus of the kinds we have considered.



An equation of Clairaut's form, $y = px+f(p)$, has a singular solution which may in particular reduce to a point. Since the equation denotes a family of straight lines, it can have no tac, cusp, or node locus.

Each of the following examples illustrates three types of loci.

EXAMPLE 1. $9p^2(2-y)^2 = 4(3-y)$.

If this equation is solved for p the solution is found by integration to be $(x-c)^2 = y^2(3-y)$. The p-discriminant relation is $(2-y)^2(3-y) = 0$ and the c-discriminant relation is $y^2(3-y) = 0$. Of the three loci $y = 0$, $y = 2$, $y = 3$, the last is the only one which satisfies the differential equation. It is the envelope of the family of curves. $y = 2$, which occurs in the p-discriminant but not in the c-discriminant relation, is a tac locus, while $y = 0$, in the c-discriminant but not in the p-discriminant relation, is a node locus.

EXAMPLE 2 $(x-y)^2(1+p^2)^3 = a^2(1+p^3)^2$

Writing this equation in the form

$$x-y = \frac{a(1+p^3)}{(1+p^2)^{\frac{3}{2}}}$$

and differentiating with regard to x we get after a slight reduction

$$1-p = 3a\frac{p(p-1)}{(1+p^2)^{\frac{5}{2}}}\frac{dp}{dx}$$

from which $1-p = 0$ or $dx = -\dfrac{3ap}{(1+p^2)^{\frac{5}{2}}}\,dp$ The solution of the latter equation is

$$x-c = a(1+p^2)^{-\frac{3}{2}}$$

or
$$1+p^2 = \frac{a^{\frac{2}{3}}}{(x-c)^{\frac{2}{3}}},$$

from which
$$p^3 = \frac{\{a^{\frac{2}{3}}-(x-c)^{\frac{2}{3}}\}^{\frac{3}{2}}}{x-c}$$

These values substituted in the given equation give the general solution

$$(x-c)^{\frac{2}{3}}+(y-c)^{\frac{2}{3}} = a^{\frac{2}{3}}$$

which denotes a set of equal 4 cusped hypocycloids with axes parallel to the coordinate axes and centres on the line $y = x$

The substitution of $p = 1$ which was obtained in the course of the work gives the singular solution $x-y = \pm\frac{1}{2}a\sqrt{2}$ Let us examine the p-discriminant relation Differentiating the original equation with regard to p we get

$$(x-y)^2(1+p^2)^2p = a^2(1+p^3)p^2$$

By multiplying the left member of the given equation by the right member of this and vice versa we obtain

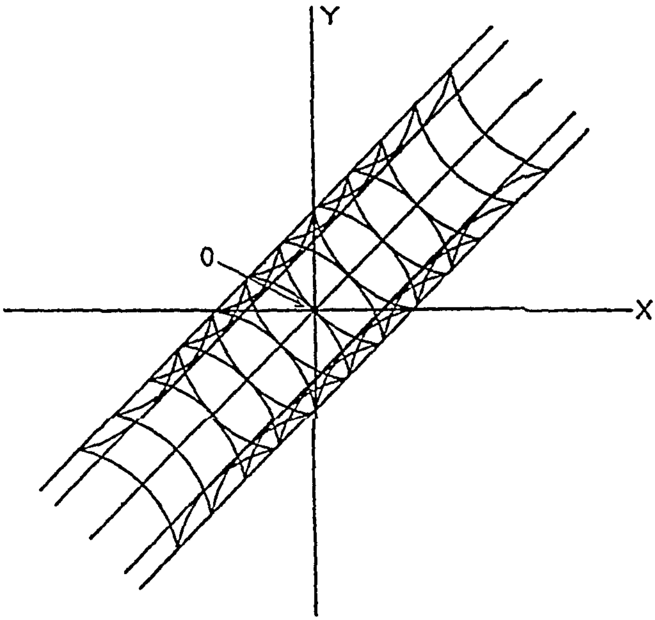
$$(x-y)^2(1+p^2)^3(1+p^3)p^2 = (x-y)^2(1+p^2)^2(1+p^3)^2p$$

from which either $x-y = 0$ $p = 0$ $p = -1$ or $p = 1$ $p = 1$ has already been considered. $p = 0$ gives on substitution $x-y = \pm a$ and $p = -1$ gives $x-y = 0$ There are thus five loci contained in the p-discriminant relation viz

$$x-y = \pm\tfrac{1}{2}a\sqrt{2} \qquad x-y = 0 \qquad x-y = \pm a$$

Of these the first two are the only ones that satisfy the dif

ferential equation. These lines are envelopes of the family of curves. It is evident from the figure that the line $x-y = 0$ is a tac locus and the lines $x-y = \pm a$ are cusp loci.



EXAMPLES ON CHAPTER IV

For each of the following equations find the general and singular solutions and any tac, cusp, or node loci that exist; also sketch the curves.

1. $y = xp+a/p$.
2. $(y-px)^2+a^2p = 0$.
3. $y-px = \sqrt{(1-p^2)}-p\cos^{-1}p$.
4. $x-y = \log p-p$.
5. $27(y-px)^2 = 4p^3$.
6. $3xy = 2px^2-2p^2$.
7. $4p^2(2p-3) = 27(y-x)$.
8. $(x-y)^2(p^2+1) = (x+yp)^2$.
9. $y-px = \dfrac{ap}{p-1}$.
10. $27y-8p^3 = 0$.
11. $p^3-4xyp+8y^2 = 0$.
12. $5y^2p^2+2xyp+x^2+4y^2-4 = 0$.
13. $x^3yp^2-(2x^2y^2-1)p+xy^3 = 0$.
14. $xyp^2-(x^2+y^2-1)p+xy = 0$. (Hint: Set $x^2 = X$, $y^2 = Y$.)
15. $(y-px)^2(1+p^2) = a^2p^2$.
16. $(16p^3-27)x = 24p^2y$.

CHAPTER V

APPLICATIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

It has already been remarked that differential equations arise in the solution of many problems in geometry mechanics and other applications of mathematics The present chapter is concerned with some examples of problems for which the mathematical formulation leads to differential equations of the first order In each problem the solution depends on a law which belongs to the subject matter of the problem considered In most cases this law is one which is readily granted requiring little technical knowledge It is important to recognize however that we are not here concerned primarily with the correctness of the law the latter is merely an assumption from which the mathematical solution proceeds

23 Laws of growth I Suppose that a variable increases at a rate which is proportional to the variable itself Such a law is strongly suggestive of the rolling of a snowball It represents approximately the rate of growth under proper conditions of certain organic substances If the variable be called x we have the differential equation $dx/dt = ax$ which gives at once the solution $x = ce^{at}$ In a specific problem the constants a and c may usually be determined from the initial conditions It will be observed that the successive values of x at the ends of equal intervals of time are in geometric progression If the constant a in the differential equation is negative the equation gives the law of decrease or decay of the variable x Physical examples of this are provided in the cases of the evaporation in the open air of moisture from a porous substance the solution of salt in an abundant supply of water etc

II A biological law of growth which agrees closely with observation in the case of many plants and young animals is that by which the rate of growth is proportional jointly to the variable itself and to its defect from a certain maximum value

If this maximum be m, the law is expressed by the differential equation $dx/dt = ax(m-x)$ in which the variables are separable.

EXAMPLE 1. If in a culture of yeast the active ferment doubles in 3 hours, by what ratio will it increase in 15 hours, assuming the first law of growth.

If x_0 is the amount of ferment at the beginning and x the amount after t hours, then $dx/dt = ax$ and $x = ce^{at}$. Now when $t = 0$, $x = x_0$, hence $c = x_0$. Also when $t = 3$, $x = 2x_0$, hence $2x_0 = x_0 e^{3a}$ and $e^{3a} = 2$. Then when $t = 15$,

$$x = x_0 e^{15a} = x_0 2^5 = 32x_0.$$

EXAMPLE 2. If a sum of A dollars bears interest at 5 per cent. continuously compounded, find the amount at the end of t years.

Let x be the amount at time t. If simple interest were reckoned on this sum for a time Δt this interest would be $0{\cdot}05x\,\Delta t$. The compound interest continuously compounded for the time Δt differs from this by an infinitesimal of higher order than Δt (this is the law on which the solution depends; it furnishes the definition of the term *continuously compounded*), $\Delta x = 0{\cdot}05x\,\Delta t + \eta$ where $\lim\limits_{\Delta t \to 0} \dfrac{\eta}{\Delta t} = 0$. From this equation, by dividing by Δt and letting Δt approach zero, we find $dx/dt = 0{\cdot}05x$, whence $x = ce^{0{\cdot}05t}$. When $t = 0$, $x = A$, hence $c = A$ and $x = Ae^{0{\cdot}05t}$. The law which is expressed by the differential equation or by its solution is sometimes called the compound interest law.

EXAMPLE 3. Assuming the temperature of the atmosphere to be constant for all altitudes, find an expression for the atmospheric pressure at any height h above the earth.

Let the pressure at height h be p and the density w. Let p_0 and w_0 be the pressure and density at the surface of the earth. By Boyle's law $\dfrac{p}{w} = \dfrac{p_0}{w_0}$ or $w = \dfrac{w_0}{p_0}p$. An increase of Δh in the altitude produces a decrease in pressure of approximately $w\,\Delta h$ (approximately since this formula neglects the change in density throughout the increment Δh). Then $\Delta p = -w\,\Delta h + \eta$ where η is an infinitesimal of higher order than Δh. This leads, as in

ex 2 to the equation $\frac{dp}{dh} = -w = -\frac{w_0}{p_0}p$ whence $p = ce^{-\frac{w_0}{p_0}h}$

When $h = 0$ $p = p_0$ Hence $c = p_0$ and $p = p_0 e^{-\frac{w_0}{p_0}h}$

EXAMPLE 4 If the natural rate of increase of the population of a country is proportional to the number of people living there and the population increases from 500 000 to 600 000 in ten years find the population as a function of the time

EXAMPLE 5 Assume that a wet porous substance in the open air loses its moisture at a rate proportional to the moisture content If in a given case half the moisture dries out in an hour when will 95 per cent of the moisture have disappeared?

EXAMPLE 6 If in the equation of law II m is ten times the initial value of x and x attains half its maximum in time t_1 show that $x = \frac{m9^{t/t_1}}{1+9^{t/t_1}}$

24 Miscellaneous problems involving rates of change

We shall not attempt to classify all the diverse problems which involve rates of change of physical quantities Problems concerning velocities pressure and temperature gradients rates of chemical reactions and rates of flow of liquids and gases are among those which lead to differential equations of the first order

EXAMPLE 1 A tank contains 100 gallons of brine in which are 50 lb of dissolved salt Fresh water is let into the tank at the rate of 2 gallons a minute The solution is kept nearly homogeneous by stirring and is led off through a tap at the rate of 1 gallon a minute How much salt does the tank contain after one half hour?

Suppose that after t minutes the tank contains x lb of salt This is dissolved in $100+t$ gallons In an interval of time Δt the amount of solution which is drawn off through the tap is Δt gallons If the concentration of the solution remained constant during this interval the amount of salt in Δt gallons would be $\frac{x}{100+t}\Delta t$ As the concentration of the solution changes very

slightly during the time Δt, the actual amount of salt which leaves the tank during Δt minutes differs from $\frac{x}{100+t}\Delta t$ by an infinitesimal of higher order than Δt. (This is the physical law which we assume.) Hence $\Delta x = -\frac{x}{100+t}\Delta t + \eta$ where $\lim \frac{\eta}{\Delta t} = 0$. Dividing by Δt and letting Δt approach zero we get $\frac{dx}{dt} = -\frac{x}{100+t}$, whence $\frac{dx}{x} = -\frac{dt}{100+t}$ and $x = \frac{c}{100+t}$. When $t = 0$, $x = 50$, hence $c = 5{,}000$ and $x = \frac{5{,}000}{100+t}$. At the end of one half-hour $t = 30$ and $x = \frac{5{,}000}{130} = 38{\cdot}5$.

Example 2. If the tank in ex. 1 contains initially 100 gallons of fresh water and a brine solution containing $\frac{1}{2}$ lb. of salt per gallon is entering at 2 gallons a minute, the solution being led off through the tap at the same rate, find the amount of salt in the tank after one half-hour.

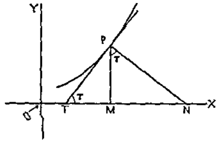
Example 3. Assume that when two reagents unite chemically the rate at which the compound is formed is at any moment proportional to the product of the amounts of the original substances untransformed. (This is called the law of mass action.) If a parts by weight of one substance combine with b parts by weight of another, find the amount of the compound formed at any time.

If A and B are the amounts of the two substances at the beginning and x the amount of compound formed at time t from the beginning, then $\frac{dx}{dt} = k\left(A - \frac{a}{a+b}x\right)\left(B - \frac{b}{a+b}x\right)$. Solve this equation with the values $A = B = 5$, $a = 1$, $b = 2$ and the boundary conditions $x = 0$ when $t = 0$, $x = 5$ when $t = 10$.

Example 4. Suppose that a hot body cools in air at a rate proportional to the difference between the temperature of the body and that of the air (Newton's law). If the air is kept at 20° C. and the body cools from 100° to 75° in 10 minutes, when will its temperature become 25°? What will be its temperature in half an hour?

25 Determination of curves from geometrical properties

(a) *Cartesian coordinates* If a property of a plane curve can be expressed in terms of an arbitrary point on it and the slope at that point the statement of this property is a differential equation of the first order The solution of the latter is the ordinary equation of a set of curves having the given property If a singular solution exists it also has the same property and may be the most significant solution of the problem It is convenient



in stating certain properties of curves to make use of the line segments shown in the figure If P $(x\ y)$ is an arbitrary point on the curve with tangent normal and ordinate drawn to the x axis the names of these segments and their values are

subnormal $\quad MN = y\tan\tau = y\dfrac{dy}{dx}$,

normal $\quad PN = y\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}$,

subtangent $\quad TM = y\cot\tau = y\dfrac{dx}{dy}$,

tangent $\quad PT = y\sqrt{\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}}$

Also we recall that the differential of arc on the curve is

$$ds = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx$$

EXAMPLE 1. Find the curve which has a constant tangent length.

The differential equation is

$$y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}=a \text{ or } \left(\frac{dx}{dy}\right)^2=\frac{a^2}{y^2}-1 \text{ or } \pm dx=\frac{\sqrt{(a^2-y^2)}}{y}d$$

Integration of this equation gives the result

$$\pm(x+c)=\sqrt{(a^2-y^2)}+a\log\frac{a-\sqrt{(a^2-y^2)}}{y}.$$

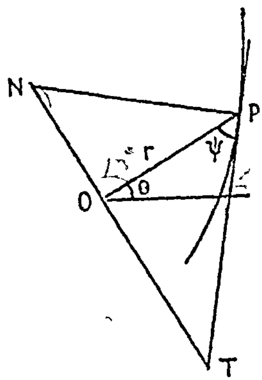
The curve obtained by giving c a fixed value is known as the tractrix.

EXAMPLE 2. Find the curve with a constant subnormal.

EXAMPLE 3. Find the curve with a constant subtangent.

EXAMPLE 4. Find the curve in which the subnormal is equal to the abscissa.

(b) *Polar coordinates.* When the equation of a curve is given in polar coordinates the segments known as the polar tangent, normal, etc., are defined by drawing through the pole a line at



right angles to the radius vector. Then in the figure $\tan\psi=r\dfrac{d\theta}{dr}$, and the values of the segments are:

$$\text{subtangent}\quad OT=r\tan\psi=r^2\frac{d\theta}{dr},$$

$$\text{tangent}\quad PT=r\sqrt{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}},$$

subnormal $ON = r\cot\psi = \dfrac{dr}{d\theta}$,

normal $PN = \sqrt{\left\{r^2+\left(\dfrac{dr}{d\theta}\right)^2\right\}}$

Also the differential of arc is $ds = \sqrt{\left\{1+r^2\left(\dfrac{d\theta}{dr}\right)^2\right\}}\,dr$

EXAMPLE 5 Determine the curve whose polar tangent at any point is k times the radius vector to that point ($k > 1$)

EXAMPLE 6 Deduce the polar equation of the family of curves in which the length of arc is proportional to the vectorial angle

26 Trajectories The term trajectory may be applied to a curve which cuts a given family of curves in any stated manner An important case is that in which the trajectory cuts every member of the family at a constant angle If this angle is a right angle the trajectory is called orthogonal, otherwise it is called oblique

(a) *Cartesian coordinates* Suppose that a given family of curves has the differential equation $f(x, y, p) = 0$ If (x, y, p) is a triple of values satisfying this equation, then through the point (x, y) there passes a curve of the family with slope p If p' be the slope of the orthogonal trajectory through the same point, then $p = -\dfrac{1}{p'}$ and p' must therefore satisfy the equation $f\left(x, y -\dfrac{1}{p'}\right) = 0$ In this equation p may now be replaced by $p = dy/dx$ and we get the differential equation of the orthogonal trajectories, $f\left(x\ y\ -\dfrac{1}{p}\right) = 0$

If a trajectory makes a constant angle α with the curves of a family given by $f(x\ y, p) = 0$, and if p' is the slope of the trajectory at the point $(x\ y)$ we have $\tan\alpha = \dfrac{p'-p}{1+p\,p}$ or $p = \dfrac{p'-\tan\alpha}{1+p\ \tan\alpha}$ By the same reasoning as in the previous case

the differential equation of the oblique trajectories is

$$f\left(x, y, \frac{p-\tan\alpha}{1+p\tan\alpha}\right) = 0.$$

(b) *Polar coordinates. Orthogonal trajectories.* Consider a family of curves whose differential equation in polar coordinates is $f(r, \theta, d\theta/dr) = 0$. At the point (r, θ) on one of these curves let ψ be the angle between the tangent to the curve and the radius vector. For the orthogonal trajectory through this point let the corresponding angle be ψ' so that $\psi' = \psi \pm \frac{1}{2}\pi$. Then $\tan\psi' = -\cot\psi = -\frac{1}{r}\frac{dr}{d\theta}$. If now $(d\theta/dr)'$ be the value of the derivative found from the equation of the orthogonal trajectory, we have $r\left(\frac{d\theta}{dr}\right)' = -\frac{1}{r}\frac{dr}{d\theta}$ and $\frac{dr}{d\theta} = -r^2\left(\frac{d\theta}{dr}\right)'$. Hence the value of $(d\theta/dr)'$ at the point (r, θ) must satisfy the equation

$$f\left\{r, \theta, -\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)'\right\} = 0.$$

If we drop the prime, which has now served its purpose, we have the differential equation of the orthogonal trajectories, $f\left(r, \theta, -\frac{1}{r^2}\frac{dr}{d\theta}\right) = 0.$

EXAMPLE 1. Find the orthogonal trajectories of a family of circles which touch a given line at a given point.

Take the given line as axis of y and the given point as origin. The family of circles has the equation $(x-a)^2+y^2 = a^2$. The differential equation is found by eliminating a. Thus $x-a+yp = 0$ or $a = x+yp$; hence $(p^2+1)y^2 = (x+yp)^2$ or $x^2-y^2+2xyp = 0$. The differential equation of the orthogonal trajectories is $x^2-y^2-\frac{2xy}{p} = 0$ or $(x^2-y^2)\,dy-2xy\,dx = 0$ or

$$\frac{x^2\,dy-2xy\,dx}{y^2}-dy = 0$$

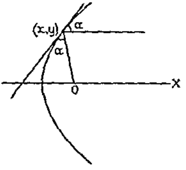
from which $\frac{x^2}{y}+y = b$ or $x^2+y^2 = by$, which denotes a set of circles touching the x-axis at the origin.

EXAMPLE 2. Find the orthogonal trajectories of the family of ellipses $x^2+2y^2 = a^2$.

EXAMPLE 3 Find the orthogonal trajectories of the family of logarithmic spirals $r = e^{a\theta}$

EXAMPLE 4 Find an equation for a set of oblique trajectories to a family of concentric circles the angle of intersection being 45°

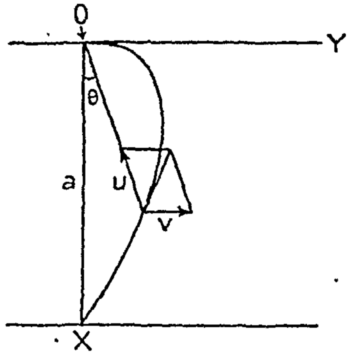
27 Determination of curves from mechanical or physical properties A wide variety of problems may be proposed in each of which a curve is required to satisfy some mechanical or physical property the mathematical statement of which is a differential equation



EXAMPLE 1 Find the shape of a plane curve such that light striking it from a fixed point source in the same plane is reflected in a parallel beam Take the origin at the point-source and the x axis in the direction of the parallel beam At $(x\ y)$ any point on the curve let the slope of the curve be p Since the incident and reflected rays make equal angles with the tangent at $(x\ y)$ we have $p = \tan\alpha = \dfrac{(y/x)-p}{1+(y/x)p} = \dfrac{y-px}{x+py}$ or $y = 2px+p^2y$ On multiplying this equation by y and setting $y^2 = v$ we get $v = x\dfrac{dv}{dx}+\dfrac{1}{4}\left(\dfrac{dv}{dx}\right)^2$ This equation is in Clairaut s form and gives the solution $y^2 = cx+\frac{1}{4}c^2$ which denotes a family of parabolas with axes along the x axis and common focus at the origin

EXAMPLE 2 A duck swims across a river heading always for

the point on the bank directly opposite her starting-point. Assuming the speed of the river to be a constant v and the duck's speed in still water a constant u, find her path.



Take the origin at the point on the bank towards which the duck swims and the axes as shown in the figure. The components parallel to the axes, of the duck's velocity, are

$$\frac{dx}{dt} = -u\cos\theta \quad \text{and} \quad \frac{dy}{dt} = v - u\sin\theta.$$

Hence $\dfrac{dy}{dx} = \dfrac{v-u\sin\theta}{-u\cos\theta}$. Now $\sin\theta = \dfrac{y}{\sqrt{(x^2+y^2)}}$, $\cos\theta = \dfrac{x}{\sqrt{(x^2+y^2)}}$.

Hence if $\dfrac{v}{u} = k$, $\dfrac{dy}{dx} = \dfrac{y-k\sqrt{(x^2+y^2)}}{x}$. On substituting $y = zx$ we get $\dfrac{dz}{\sqrt{(1+z^2)}} + k\dfrac{dx}{x} = 0$ from which

$$\log\{z+\sqrt{(1+z^2)}\}+k\log x = \log c,$$

or $z+\sqrt{(1+z^2)} = cx^{-k}$. When $x = a$, $y = 0$ and $z = 0$, hence $c = a^k$ and $z+\sqrt{(1+z^2)} = (x/a)^{-k}$. Solving for z we get

$$z = \frac{1}{2}\left\{\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right\}$$

or

$$y = \frac{x}{2}\left\{\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right\} = \frac{a}{2}\left\{\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right\}.$$

It is interesting to examine this solution for different values of k. If $k < 1$ the curve passes through the origin. That is, if the speed of the river is less than the duck's speed in still water

she reaches her destination If $k = 1$ $y = \frac{1}{2}a\{1-(x/a)^2\}$ which cuts the y axis at $(0, \frac{1}{2}a)$ That is when $u = v$ the duck reaches a point on the bank distant $\frac{1}{2}a$ from her destination If $k > 1$ the y axis is asymptotic to the curve Hence when $v > u$ the duck is carried down stream without reaching the opposite bank

EXAMPLES ON CHAPTER V

1 Determine the curve in which the length of the tangent is equal to the intercept made by the tangent on the x-axis

2 Determine the curve in which the perpendicular distance from the origin on any tangent is equal to the abscissa of the point of contact

3 Determine the curve whose tangent cuts off from the coordinate axes intercepts whose sum is constant

4 Determine the curve whose tangent makes with the coordinate axes a triangle of constant area

5 Determine the curve whose polar subnormal is n times the polar subtangent

6 Find the orthogonal trajectories of the conics $x^2+ay^2 = 1$ and illustrate by a figure

7 Find the orthogonal trajectories of the set of parabolas $y^2 = ax$

8 Find the orthogonal trajectories of the hyperbolas $xy = k^2$

9 Find the orthogonal trajectories of the circles $x^2+(y-a)^2 = 1+a^2$ and illustrate by a figure

10 Determine the curve for which the sum of the perpendiculars from two fixed points to any tangent is constant.

11 Determine the curve for which the rectangle contained by the perpendiculars from two fixed points to any tangent is constant

12 Determine the curve whose normal at any point makes equal angles with the radius vector and the initial line

13 Find the orthogonal trajectories of the spirals $r\theta = a$

14 Find the orthogonal trajectories of the cardioids $r = a(1-\cos\theta)$

15 Find the curve for which the length of the tangent intercepted between the coordinate axes is constant

16. Show that the family of parabolas $y^2 = 4a(x+a)$ is self-orthogonal

17 Show that a family of confocal ellipses and hyperbolas is self orthogonal

18 Suppose that a solid sphere of salt dissolves in running water at a rate proportional to the surface area of the sphere If half the salt dissolves in 15 minutes in what time will it be all dissolved?

19 Water is poured slowly into a cask containing vinegar and the mixture (assumed homogeneous) is drawn off through a tap at the same rate What percentage of vinegar does the cask contain when three times the volume of the cask has passed through the tap?

20 A tank contains 100 gallons of brine with 25 lb of dissolved salt

A solution containing $\frac{1}{2}$ lb. of salt per gallon enters the tank at the rate of 2 gallons a minute and the resulting solution (assumed homogeneous) is drawn off at the rate of 1 gallon a minute. Find the amount of salt in the tank at the end of one half-hour.

21. A 12-quart pail full of maple sap is exposed to a rainfall of 2 inches, the diluted solution overflowing. Assuming that the sugar solution is always homogeneous, find the ratio of the amounts of sugar in the pail before and after the rain. Take the diameter of the top of the pail to be 1 foot.

22. Each of two vessels contains 10 gallons of brine with 5 lb. of dissolved salt. Fresh water runs into the first at the rate of 1 gallon a minute. The resulting solution is drawn off from the first and runs into the second vessel at the same rate. Again the solution runs out of the second vessel at 1 gallon a minute. Assuming that each solution is kept homogeneous by stirring, find how much salt the second vessel contains after 10 minutes.

23. The air in a room is changed slowly by ventilation, warm air at a temperature of 40° C. being admitted. If the room air was originally at 10° C., what will be its temperature when a volume of warm air sufficient to fill the room has been admitted? Assume the air throughout the room to be of uniform temperature at any time.

24. A mass of 10 kg. of brass (sp. heat 0·09), at a temperature of 100° C., is placed in a vat containing 10 litres of water at 0° C. In one minute the temperature of the brass has fallen to 50° C. Assuming that no heat escapes to the walls of the containing vessel or to the surrounding air, find the temperature of the brass as a function of the time.

25. Find the path of a projectile which has an initial velocity v at an angle α with the horizontal.

26. Find the shape of a curve such that rays of light striking it from a fixed point source in the plane are all reflected to a second point of the plane.

27. A man swims across a river, always heading for the nearest point of the opposite bank. Find his path, supposing the velocity of the river proportional to the distance from the nearer bank.

28. Assuming that the temperature (in absolute units) of the atmosphere falls off uniformly from T_0 at the surface of the earth to 0 at height H, at which the atmosphere disappears, show that the pressure at height h is given by $p = p_0(1-h/H)^{H/RT_0}$, where p_0 is the pressure at the earth's surface and R is constant.

CHAPTER VI

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

28 The general linear equation Composition of solutions The general linear equation of the nth order is

$$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_{n-1}\frac{dy}{dx}+P_ny=Q, \qquad (1)$$

in which the P's and Q are functions of x. Both on account of the extent to which their theory has been developed and of the frequency of their occurrence in mechanical and physical problems, linear equations constitute the most important single class of ordinary differential equations. No method is known which will solve all linear equations. Solvable cases are, however, obtained by specializing the coefficients P and the right member Q.

The linear equation in which Q is replaced by zero may be called the *reduced equation* of (1). The following principles, which may be verified by direct substitution, serve as a guide in searching for general solutions of linear equations.

(*a*) If $y=f(x)$ is a solution of the reduced equation, then $y=cf(x)$, where c is constant, is also a solution.

(*b*) If $y=f_1(x)$ and $y=f_2(x)$ are two solutions of the reduced equation, then $y=f_1(x)+f_2(x)$ is a solution. It follows that $y=c_1f_1(x)+c_2f_2(x)$ is a solution. Moreover, if by any method n linearly independent solutions† $f_1(x)$, $f_2(x)$, …, $f_n(x)$ have been obtained, then $y=c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)$ is a solution containing n arbitrary constants and is therefore the general solution of the reduced equation.

(*c*) If $y=\phi(x)$ is a solution of (1) containing no arbitrary constants and $y=c_1f_1(x)+c_2f_2(x)+\cdots+c_nf_n(x)$ is the general solution of its reduced equation, then

$$y=\phi(x)+c_1f_1(x)+\cdots+c_nf_n(x) \qquad (2)$$

† The functions f_1, f_2, …, f_n are said to be linearly dependent if a set of constants a_1, a_2, …, a_n, not all zero, exists such that $a_1f_1+a_2f_2+\cdots+a_nf_n=0$. If no such set of constants exists the functions are linearly independent.

is the general solution of (1). For the substitution of $\phi(x)$ in the left member of (1) reduces it to Q and the substitution of $c_1 f_1(x)+\ldots+c_n f_n(x)$ reduces it to zero. Hence (2) is a solution of (1), and since it contains n arbitrary constants it is the general solution. In this solution of (1) the function $\phi(x)$ is called a *particular integral* and $c_1 f_1(x)+\ldots+c_n f_n(x)$ the *complementary function.*

29. Linear equations with constant coefficients and right member zero. The general equation of this type is

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}\frac{dy}{dx}+a_n y = 0. \qquad (1)$$

The substitution of $y = e^{mx}$ in the left member of this equation gives $e^{mx}(m^n+a_1 m^{n-1}+a_2 m^{n-2}+\ldots+a_{n-1} m+a_n)$. Consequently $y = e^{mx}$ is a solution of (1) provided m is a root of the algebraic equation $m^n+a_1 m^{n-1}+\ldots+a_{n-1} m+a_n = 0$, which is called the *auxiliary equation.* If the n roots of this equation are $m_1, m_2, \ldots, m_n$ and are all distinct, then the general solution of the differential equation is

$$y = c_1 e^{m_1 x}+c_2 e^{m_2 x}+\ldots+c_n e^{m_n x}.$$

EXAMPLE 1. $\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-6y = 0.$

The auxiliary equation is $m^2-m-6 = 0$ which has the roots 3 and -2. The general solution is therefore $y = c_1 e^{3x}+c_2 e^{-2x}$.

EXAMPLE 2. $\dfrac{d^3y}{dx^3}-13\dfrac{dy}{dx}+12y = 0.$

EXAMPLE 3. $2\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+y = 0.$

30. The case of equal roots of the auxiliary equation. If two roots m_1 and m_2 of the auxiliary equation are equal, then the solution of the differential equation corresponding to these equal roots is $c_1 e^{m_1 x}+c_2 e^{m_1 x} = (c_1+c_2)e^{m_1 x}$ which contains essentially only one arbitrary constant c_1+c_2. It follows that in this case the solution obtained in the preceding paragraph contains fewer than n arbitrary constants and is therefore not

the general solution The result obtained in the following example indicates the nature of the solution in such a case A second method is given in art 33

EXAMPLE $\dfrac{d^2y}{dx^2}-2a\dfrac{dy}{dx}+a^2y=0$

This equation has the auxiliary equation $(m-a)^2=0$ and hence $y=ce^{ax}$ is one solution Let us replace the constant c by a function $v(x)$ and inquire whether v can be determined other than a constant so that $y=ve^{ax}$ satisfies the differential equation † If $y=ve^{ax}$ we have

$$\frac{dy}{dx}=e^{ax}\frac{dv}{dx}+ave^{ax}$$

and
$$\frac{d^2y}{dx^2}=e^{ax}\frac{d^2v}{dx^2}+2ae^{ax}\frac{dv}{dx}+a^2ve^{ax}$$

Hence $\dfrac{d^2y}{dx^2}-2a\dfrac{dy}{dx}+a^2y=e^{ax}\dfrac{d^2v}{dx^2}$ and the equation is satisfied if $d^2v/dx^2=0$ or if $v=c_1+c_2x$ This gives $y=(c_1+c_2x)e^{ax}$ as the general solution of the given equation

31 The case of imaginary roots of the auxiliary equation

It remains to consider the form of solution corresponding to imaginary roots of the auxiliary equation As this equation has real coefficients imaginary roots if such exist must occur in pairs of the form $\alpha\pm i\beta$ where $i=\sqrt{-1}$ In this case the expressions $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ formally satisfy the differential equation but we are faced with the problem of giving a meaning to these exponential functions with imaginary exponents We shall avoid this problem by solving directly the second-order equation whose auxiliary equation has the roots $\alpha\pm i\beta$ The extension to equations of higher order will be made in art 32

Consider first the differential equation $\dfrac{d^2y}{dx^2}+\beta^2y=0$ for which the auxiliary equation $m^2+\beta^2=0$ has the imaginary roots $\pm i\beta$ This equation becomes an identity on the substitu

† This is a simple example of the method of variation of parameters Cf art 51

tion of $\cos\beta x$ or of $\sin\beta x$ for y. Hence the general solution is $y = c_1\cos\beta x + c_2\sin\beta x$. Take next the equation

$$\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + (\alpha^2+\beta^2)y = 0 \tag{1}$$

for which the auxiliary equation is $m^2 - 2\alpha m + \alpha^2 + \beta^2 = 0$ with roots $\alpha \pm i\beta$. This equation is transformed to the former case by the substitution of $e^{\alpha x}z$ for y. For then

$$y = e^{\alpha x}z, \qquad \frac{dy}{dx} = e^{\alpha x}\left(\frac{dz}{dx} + \alpha z\right),$$

$$\frac{d^2y}{dx^2} = e^{\alpha x}\left(\frac{d^2z}{dx^2} + 2\alpha\frac{dz}{dx} + \alpha^2 z\right),$$

whence $$\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + (\alpha^2+\beta^2)y \equiv e^{\alpha x}\left(\frac{d^2z}{dx^2} + \beta^2 z\right)$$

and (1) transforms into $\dfrac{d^2z}{dx^2} + \beta^2 z = 0$. Since this equation has the general solution $z = A\cos\beta x + B\sin\beta x$, it follows that (1) has the general solution $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$. An equivalent form for the solution is $y = ke^{\alpha x}\cos(\beta x + \gamma)$ with the arbitrary constants k and γ. It is verified by expansion of $\cos(\beta x + \gamma)$ that the constants of the two solutions are connected by the relations $A = k\cos\gamma$, $B = -k\sin\gamma$.

EXAMPLE 1. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0.$

The auxiliary equation is $m^2 - 2m + 2 = 0$ which has the roots $1 \pm i$. Hence the solution may be written in either of the forms $y = e^x(A\cos x + B\sin x)$ or $y = ke^x\cos(x+\gamma)$.

EXAMPLE 2. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 4y = 0.$

EXAMPLE 3. $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 61y = 0.$

32. The operators D and $f(D)$. In the further study of linear equations great advantage is found to result from using the symbols D, D^2, D^3, etc., to stand for the differential operators d/dx, d^2/dx^2, d^3/dx^3, etc. The linear equation of the nth order

with constant coefficients and right member a function of x may then be written

$$(D^n+a_1 D^{n-1}+\cdots+a_{n-1}D+a_n)y = Q$$

The operator $D^n+a_1 D^{n-1}+\cdots+a_n$ has the *form* of a polynomial and may be denoted by $f(D)$. The differential equation is then $f(D)y = Q$.

The usefulness of the symbolic operators D and $f(D)$ results from the circumstance that in many operations D behaves as if it were an algebraic quantity. This is due to the fact that the symbol obeys the following laws of algebra:

(1) it is distributive over the terms of a sum

$$D(u+v) = Du+Dv$$

(2) it is commutative with constants $D(cu) = c\,Du$

(3) it obeys the index law $D^m(D^n u) = D^{m+n}u.$

It is to be noted that the property which differentiates it from an algebraic quantity is that it is not commutative with variables, that is, Duv is not equal to uDv. It is for this reason that the chief usefulness of the symbol arises in the study of equations with constant coefficients.

A first result of these properties is that if the polynomial $f(D)$ be separated into factors $(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)$ which may be written in any order, the two symbols regarded as operators have the same meaning. For the change from $(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)y$ to $f(D)y$ requires the use of only the three laws enumerated above.

Suppose now that $f(D)$ is resolved into two factors so that $f(D)y = f_1(D)\phi(D)y$. If y is such a function of x that $\phi(D)y \equiv 0$, it follows that $f(D)y \equiv 0$. Thus a solution of $f(D)y = 0$ is obtained from any factor of $f(D)$. If such a factor is

$$D^2-2\alpha D+\alpha^2+\beta^2$$

the corresponding solution of $f(D)y = 0$ is

$$e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$

The result of art. 31 for second-order equations is therefore extended to an equation of any order in which the auxiliary equation has a pair of imaginary roots.

33. The device of factoring the operator $f(D)$ furnishes an alternative method of treating the equation $f(D)y = 0$ in case the auxiliary equation has repeated roots. The method will be illustrated by solving the equation $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = 0$ which was solved by a different method in art. 30. In symbolic form the equation is $(D-a)^2y = 0$ or $(D-a)(D-a)y = 0$. Denote $(D-a)y$ by v. The equation is then $(D-a)v = 0$ from which $v = ce^{ax}$. Hence y satisfies the equation $(D-a)y = ce^{ax}$, a linear equation of the first order which gives $y = e^{ax}(c_1x+c_2)$. By a repetition of this process the student should show that the equation $(D-a)^3y = 0$ has the solution $y = e^{ax}(c_1x^2+c_2x+c_3)$ and that $(D-a)^ny = 0$ has the solution

$$y = e^{ax}(c_1x^{n-1}+c_2x^{n-2}+\ldots+c_n).$$

EXAMPLE 1. $(D-b)(D-a)^2y = 0$.

Setting $(D-a)y = v$ we have $(D-b)(D-a)v = 0$ whence $v = c_1e^{bx}+c_2e^{ax}$. Then $(D-a)y = c_1e^{bx}+c_2e^{ax}$ gives

$$y = e^{ax}\left\{\int e^{-ax}(c_1e^{bx}+c_2e^{ax})\,dx + c_3\right\} = C_1e^{bx}+(c_2x+c_3)e^{ax}$$

where $C_1 = c_1/(b-a)$.

EXAMPLE 2. $(D-b)^2(D-a)^2y = 0$.

EXAMPLE 3. $(D^2+1)^2y = 0$.

The auxiliary equation has the roots $\pm i$, each repeated. By art. 31 $y = \cos x$ and $y = \sin x$ are solutions. The above results suggest the general solution $y = (c_1+c_2x)\cos x+(c_3+c_4x)\sin x$, which may be verified.

34. The inverse operator $\frac{1}{f(D)}$. It has already been pointed out that the general solution of the equation $f(D)y = Q$ consists of two parts, a particular integral which contains no arbitrary constants and the complementary function which is the general solution of the reduced equation. Methods of finding the complementary function having now been considered, it remains to devise methods of finding a particular integral. A particular integral may be symbolized by $y = \frac{1}{f(D)}Q$. The

symbol $\frac{1}{f(D)}$ is then a new operator defined by the fact that $\frac{1}{f(D)}Q$ is such a function P that $Q = f(D)P$ This suggests the analogy of other inverse operations Thus if $Q = \sin P$ then $P = \sin^{-1}Q$ if $Q = e^P$ then $P = \log Q$, and if $Q = dP/dx$ then $P = \int Q\,dx$ This last illustration is indeed a particular case of the operations under discussion, for if $Q = DP$, then $P = \frac{1}{D}Q$ and the symbol $\frac{1}{D}$ or D^{-1} denotes integration Furthermore, while the result of the operation $f(D)P$ is to give a definite function Q, the result of the operation $\frac{1}{f(D)}Q$ is indefinite in much the same way as ordinary integration is indefinite We have by definition $f(D)\frac{1}{f(D)}Q = Q$ On the other hand, $\frac{1}{f(D)}f(D)Q$ is such a function P that $f(D)P = f(D)Q$, and two different functions P and Q may satisfy this relation Thus, for example, $D^3x^4 = D^3(x^4+c_1x^2+c_2x+c_3)$ where the c's are arbitrary constants When therefore we write $\frac{1}{f(D)}f(D)Q = Q$ we are to understand that the right member of this equation is one of the many functions denoted by the left member

35. The particular integral of $f(D)y = Q$ General methods.

First method The linear equation of the first order is $(D-a)y = Q$ Its solution has been found to be

$$y = e^{ax}\int e^{-ax}Q\,dx,$$

from which follows the important formula

$$\frac{1}{D-a}Q = e^{ax}\int e^{-ax}Q\,dx$$

If now the operator $f(D)$ be expressed in factors, we have $f(D)y = (D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)y = Q$ Hence

$$(D-\alpha_2)\cdots(D-\alpha_n)y = \frac{1}{D-\alpha_1}Q = e^{\alpha_1 x}\int e^{-\alpha_1 x}Q\,dx$$

Also

$$(D-\alpha_3)\dots(D-\alpha_n)y = \frac{1}{D-\alpha_2}e^{\alpha_1 x}\int e^{-\alpha_1 x}Q\,dx$$
$$= e^{\alpha_2 x}\int e^{(\alpha_1-\alpha_2)x}\int e^{-\alpha_1 x}Q\,(dx)^2.$$

By successive operations of this kind we obtain finally

$$y = e^{\alpha_n x}\int e^{(\alpha_{n-1}-\alpha_n)x}\int\dots\int e^{-\alpha_1 x}Q\,(dx)^n.$$

Second method. Suppose that, thinking of D as an algebraic quantity, we decompose $1/f(D)$ into its partial fractions. Thus

$$\frac{1}{f(D)} = \frac{A_1}{D-\alpha_1}+\frac{A_2}{D-\alpha_2}+\dots+\frac{A_n}{D-\alpha_n}, \tag{1}$$

so that

$$1 = A_1(D-\alpha_2)\dots(D-\alpha_n)+A_2(D-\alpha_1)(D-\alpha_3)\dots(D-\alpha_n)+$$
$$+\dots+A_n(D-\alpha_1)\dots(D-\alpha_{n-1}). \tag{2}$$

Each member of (1) may now be interpreted as an operator. The question is, have they as operators the same meaning or is the following equation true:

$$\frac{1}{f(D)}y = \left(\frac{A_1}{D-\alpha_1}+\dots+\frac{A_n}{D-\alpha_n}\right)y\,?$$

This will be the case if

$$y = f(D)\left(\frac{A_1}{D-\alpha_1}+\dots+\frac{A_n}{D-\alpha_n}\right)y. \tag{3}$$

Since $f(D)$ may be expressed in factors which may be written in any order, the right member of (3) is

$$\{A_1(D-\alpha_2)\dots(D-\alpha_n)+A_2(D-\alpha_1)(D-\alpha_3)\dots(D-\alpha_n)+$$
$$+\dots+A_n(D-\alpha_1)\dots(D-\alpha_{n-1})\}y.$$

The polynomial in the bracket reduces to 1 by the use of (2), and by virtue of the laws governing the operator D this expression has the value y. The particular integral now takes the form†

$$A_1 e^{\alpha_1 x}\int e^{-\alpha_1 x}Q\,dx + A_2 e^{\alpha_2 x}\int e^{-\alpha_2 x}Q\,dx +$$
$$+\dots+A_n e^{\alpha_n x}\int e^{-\alpha_n x}Q\,dx.$$

† The formulae of this section may be applied formally even when the roots of $f(D) = 0$ are imaginary. The resulting solutions, however, involve functions

EXAMPLE 1 $\quad (D^2-4D+3)y = e^{4x}$

This equation may be written $(D-1)(D-3)y = e^{4x}$ By the first method

$$(D-3)y = \frac{1}{D-1}e^{4x} = e^x \int e^{-x}e^{4x}\,dx = e^x\frac{e^{3x}}{3} = \frac{e^{4x}}{3}$$

Then $\quad \dfrac{1}{D-3}\,\dfrac{e^{4x}}{3} = \frac{1}{3}e^{3x}\int e^{-3x}e^{4x}\,dx = \frac{1}{3}e^{3x}e^x = \dfrac{e^{4x}}{3}.$

By the second method

$$y = \frac{1}{(D-1)(D-3)}e^{4x} = \frac{1}{2}\left(\frac{1}{D-3} - \frac{1}{D-1}\right)e^{4x},$$

which gives the same result The complementary function is found in the usual way

EXAMPLE 2 $\quad (2D^2+5D+2)y = 5$

EXAMPLE 3 $\quad (D^2-4)y = x$

The general methods of the present article may be employed to determine a particular integral, no matter what may be the function Q in the right member of the equation There are, however, many cases in which a particular integral may be found more readily than by performing the integrations involved in the general methods These cases will now be considered

36 $\quad f(D)y = e^{ax} \qquad$ Since $\quad De^{ax} = ae^{ax}, \quad D^2e^{ax} = a^2e^{ax},\ ,$ $D^ne^{ax} = a^ne^{ax}$, hence $f(D)\,e^{ax} = f(a)\,e^{ax}$ Dividing this equation by the constant $f(a)$ which we assume for the present is not zero we get $\dfrac{1}{f(a)}f(D)\,e^{ax} = e^{ax}$ Since $f(D)$ is commutative with

with imaginary arguments From these functions real solutions may be derived by the methods of the theory of functions of a complex variable Real solutions may however be obtained directly Thus in the equation $\{(D-\alpha)^2+\beta^2\}y = Q$

$$f(D) - (D-\alpha+i\beta)(D-\alpha-i\beta)$$

The solution of this equation may be obtained from the formula

$$\frac{1}{(D-\alpha)^2+\beta^2}Q = \frac{e^{\alpha x}}{\beta}\left[\sin\beta x\int Qe^{-\alpha x}\cos\beta x\,dx - \cos\beta x\int Qe^{-\alpha x}\sin\beta x\,dx\right]$$

This formula may be verified directly It is conveniently derived by the method of variation of parameters. Cf art 51

a constant† this may be written $f(D)\dfrac{e^{ax}}{f(a)} = e^{ax}$, whence it follows that $\dfrac{e^{ax}}{f(a)} = \dfrac{1}{f(D)}e^{ax}$. A particular integral of the given equation in case $f(a) \neq 0$ is therefore $y = e^{ax}/f(a)$.

If $f(a) = 0$ the polynomial $f(D)$ contains the factor $D-a$, repeated, it may be, m times. Then $f(D) = (D-a)^m\phi(D)$ where $\phi(a) \neq 0$. In this case $\phi(D)(D-a)^m y = e^{ax}$. Hence

$$(D-a)^m y = \frac{1}{\phi(D)}\,e^{ax} = \frac{e^{ax}}{\phi(a)}$$

and
$$y = \frac{1}{(D-a)^m}\,\frac{e^{ax}}{\phi(a)} = \frac{1}{\phi(a)}\,\frac{1}{(D-a)^m}\,e^{ax},$$

which may be evaluated by the first method of art. 35 or by the method of art. 37.

EXAMPLE 1. $(D^2-4)y = (1+e^x)^2$.

A particular integral is $\dfrac{1}{D^2-4}(1+2e^x+e^{2x})$. Now

$$\frac{1}{D^2-4}\,1 = \frac{1}{D^2-4}\,e^{0x} = \frac{e^{0x}}{-4} = -\tfrac{1}{4}.$$

Again $\dfrac{1}{D^2-4}\,2e^x = \dfrac{2e^x}{1-4} = -\tfrac{2}{3}e^x$. Also

$$\frac{1}{D^2-4}\,e^{2x}$$
$$= \frac{1}{D-2}\,\frac{1}{D+2}\,e^{2x} = \frac{1}{D-2}\,\frac{e^{2x}}{4} = \tfrac{1}{4}e^{2x}\int e^{-2x}e^{2x}\,dx = \tfrac{1}{4}e^{2x}x.$$

The complete solution is $y = \dfrac{xe^{2x}}{4} - \tfrac{2}{3}e^x - \tfrac{1}{4} + c_1\,e^{2x} + c_2\,e^{-2x}$.

EXAMPLE 2. $(3D^2+10D-8)y = 7e^{3x}$.

EXAMPLE 3. $(D^3-a^3)y = e^{bx}$.

† The fact that $f(D)$ is commutative with constants is apparent. That $1/f(D)$ is also commutative with constants may not be so obvious. Suppose $y = \dfrac{1}{f(D)}cQ$. Then $f(D)y = cQ$. Hence $\dfrac{1}{c}f(D)y = Q$ and $f(D)\dfrac{y}{c} = Q$ whence $\dfrac{y}{c} = \dfrac{1}{f(D)}Q$ and $y = c\dfrac{1}{f(D)}Q$.

37. $f(D)y = e^{ax}V$ **where** V **is a function of** x We have

$$De^{ax}V = e^{ax}DV + ae^{ax}V = e^{ax}(D+a)V$$

Also $D^2e^{ax}V = e^{ax}(D^2+2Da+a^2)V$ We shall write the operator $D^2+2Da+a^2$ in the abridged form $(D+a)^2$ and similarly for higher powers of $D+a$ Then $D^2e^{ax}V = e^{ax}(D+a)^2V$ In the same way it is found that $D^3e^{ax}V = e^{ax}(D+a)^3V$, and the general result $D^ne^{ax}V = e^{ax}(D+a)^nV$ is easily established by mathematical induction By adding together terms of the form $a_mD^me^{ax}V$ it follows that

$$f(D)e^{ax}V = e^{ax}f(D+a)V \tag{1}$$

The result of the inverse operation $1/f(D)$ on the product $e^{ax}V$ is now obtained by the following device From (1)

$$e^{ax}V = \frac{1}{f(D)}e^{ax}f(D+a)V \tag{2}$$

Since V is an arbitrary function of x $f(D+a)V$ is also an arbitrary function Let it be denoted by V_1 $f(D+a)V = V_1$ Then $V = \frac{1}{f(D+a)}V_1$ whence (2) becomes $e^{ax}\frac{1}{f(D+a)}V_1 = \frac{1}{f(D)}e^{ax}V_1$ This relation being true for an arbitrary function V_1, we may now replace V_1 by V Hence $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$ This formula is useful in case the evaluation of $\frac{1}{f(D+a)}V$ is more readily accomplished than that of $\frac{1}{f(D)}e^{ax}V$

The exceptional case of art 36 may now be considered Suppose that $V = 1$ and that $f(D) = (D-a)^m$ Then

$$\frac{1}{(D-a)^m}e^{ax} = e^{ax}\frac{1}{D^m}1 = e^{ax}\int \cdots \int (dx)^m = e^{ax}\frac{x^m}{m!}$$

which disposes of this exceptional case

EXAMPLE 1 $(D^2-2D-3)y = e^{3x}$

The complementary function is $c_1e^{-x}+c_2e^{3x}$ For the particular integral $(D+1)(D-3)y = e^{3x}$

Hence $$(D-3)y = \frac{1}{D+1}e^{3x} = \frac{e^{3x}}{4}$$

and $$y = \frac{1}{4}\frac{1}{D-3}e^{3x} = \tfrac{1}{4}e^{3x}\frac{1}{D}1 = \tfrac{1}{4}e^{3x}x.$$

EXAMPLE 2. $(D^2+4D+4)y = e^{-2x}\sin x$.

EXAMPLE 3. $(D^3-1)y = e^x$.

EXAMPLE 4. $(D^2-3D+2)y = xe^{2x}$.

38. $f(D)y = \sin ax$ or $\cos ax$. Proceeding as in the previous cases we find
$$D\sin ax = a\cos ax,$$
$$D^2\sin ax = -a^2\sin ax.$$
It follows that
$$D^4\sin ax = (-a^2)^2\sin ax,$$
$$D^6\sin ax = (-a^2)^3\sin ax$$
and generally
$$D^{2m}\sin ax = (-a^2)^m\sin ax.$$
Consequently, if $F(D^2)$ is a polynomial in D^2,
$$F(D^2)\sin ax = F(-a^2)\sin ax.$$
Similarly $F(D^2)\cos ax = F(-a^2)\cos ax$.

The results of the inverse operation $1/F(D^2)$ on $\sin ax$ and $\cos ax$ are now deduced as follows on the assumption that $F(-a^2)$ is not zero. $\frac{1}{F(-a^2)}F(D^2)\sin ax = \sin ax$. Hence $F(D^2)\,\frac{\sin ax}{F(-a^2)} = \sin ax$ and $\frac{\sin ax}{F(-a^2)} = \frac{1}{F(D^2)}\sin ax$. In the same manner $\frac{1}{F(D^2)}\cos ax = \frac{\cos ax}{F(-a^2)}$.

These formulae give a particular integral in case $f(D)$ has the form $F(D^2)$, i.e. it contains only even powers of D, and $F(-a^2) \neq 0$.

The method of finding a particular integral in case $f(D)$ contains odd as well as even powers of D is illustrated in the following example:
$$(D^3+D^2+D+6)y = \cos 2x.$$

A particular integral is

$$\frac{1}{(D^3+D)+(D^2+6)}\cos 2x$$
$$= \{(D^3+D)-(D^2+6)\}\frac{1}{(D^3+D)^2-(D^2+6)^2}\cos 2x$$

(the justification of this step is left to the student)

$$= (D^3+D-D^2-6)\frac{\cos 2x}{-4(-4+1)^2-(-4+6)^2}$$
$$= -\frac{1}{40}(D^3+D-D^2-6)\cos 2x = -\frac{1}{40}(6\sin 2x-2\cos 2x)$$

The following variation of the method is sometimes shorter

$$\frac{1}{D^3+D^2+D+6}\cos 2x = \frac{1}{D(D^2+1)+D^2+6}\cos 2x$$
$$= \frac{1}{D(-4+1)-4+6}\cos 2x = \frac{1}{-3D+2}\cos 2x = \frac{2+3D}{4-9D^2}\cos 2x$$
$$= (2+3D)\frac{\cos 2x}{4+9\cdot 4} = \frac{1}{40}(2\cos 2x-6\sin 2x)$$

In the exceptional case in which $F(-a^2)=0$ the foregoing method fails to give a particular integral A method of handling this case is illustrated in the following example

$$\frac{d^2y}{dx^2}+a^2y = \cos ax \tag{1}$$

If the right member be replaced by $\cos(a+h)x$, $h \neq 0$, the equation is no longer exceptional,

$$\frac{d^2y}{dx^2}+a^2y = \cos(a+h)x \tag{2}$$

A particular integral of this equation is

$$\frac{1}{D^2+a^2}\cos(a+h)x = \frac{\cos(a+h)x}{-(a+h)^2+a^2} = \frac{\cos(a+h)x}{-2ah-h^2}$$

By use of the theorem of mean value of the differential calculus we find

$$\cos(a+h)x = \cos ax - hx\sin(a+\theta h)x, \qquad 0<\theta<1$$

Hence the particular integral of (2) becomes

$$\frac{\cos ax - hx\sin(a+\theta h)x}{-2ah-h^2}$$

The term $\dfrac{\cos ax}{-2ah-h^2}$ is included in the term $A\cos ax$ which is part of the complementary function of (2). It may therefore be omitted from the particular integral and we obtain the new particular integral

$$\frac{-hx\sin(a+\theta h)x}{-2ah-h^2} = \frac{-x\sin(a+\theta h)x}{-2a-h}.$$

This is a particular integral of (2) for all values of h other than zero. If we assume that the solution is a continuous function of h when $h = 0$, we find a particular integral of (1) to be $\dfrac{x\sin ax}{2a}$. This result is embodied in the formula

$$\frac{1}{D^2+a^2}\cos ax = \frac{x\sin ax}{2a}$$

which may be verified directly. The student should develop for himself the corresponding formula

$$\frac{1}{D^2+a^2}\sin ax = -\frac{x\cos ax}{2a}.$$

EXAMPLE 1. $(D^4-5D^2+4)y = \sin\frac{1}{2}x$.

EXAMPLE 2. $(D^2+D+1)y = \cos x$.

EXAMPLE 3. $(D^2+4)(D^2+9)y = 20\sin 2x-30\cos 3x$.

39. $f(D)y = x^m$, **m a positive integer.** We consider first the simple case $(D-a)y = x^m$. A particular integral is

$$y = \frac{1}{D-a}x^m = e^{ax}\int e^{-ax}x^m\,dx.$$

Integration by parts gives the formula

$$\int e^{-ax}x^r\,dx = -\frac{x^r e^{-ax}}{a}+\frac{r}{a}\int x^{r-1}e^{-ax}\,dx.$$

By successive applications of this formula we find

$$\frac{1}{D-a}x^m = e^{ax}\left[-\frac{x^m e^{-ax}}{a}-\frac{m}{a}\frac{x^{m-1}e^{-ax}}{a}-\right.$$
$$\left.-\frac{m(m-1)}{a^2}\frac{x^{m-2}e^{-ax}}{a}-\dots-\frac{m(m-1)\dots 2.1}{a^m}\frac{e^{-ax}}{a}\right]$$

$$= -\frac{1}{a}\left[x^m + \frac{mx^{m-1}}{a} + \frac{m(m-1)x^{m-2}}{a^2} + \dots + \frac{m(m-1)\dots 2\cdot 1}{a^m}\right]$$

It is important to observe that this same result is obtained by expanding the operator by the binomial theorem and operating on x^m by the successive powers of D. Thus

$$\frac{1}{D-a}x^m = (D-a)^{-1}x^m = -\frac{1}{a}\left(1-\frac{D}{a}\right)^{-1}x^m$$

$$= -\frac{1}{a}\left(1+\frac{D}{a}+\frac{D^2}{a^2}+\frac{D^3}{a^3}+\dots\right)x^m$$

$$= -\frac{1}{a}\left(x^m + \frac{mx^{m-1}}{a} + \frac{m(m-1)x^{m-2}}{a^2} + \dots + \frac{m(m-1)\dots 2\cdot 1}{a^m}\right)$$

Consider now the general case in which $y = \dfrac{1}{f(D)}x^m$. If the operator $1/f(D)$ be separated into its partial fractions the foregoing method will apply to each fraction. The result is equivalent to that obtained by expanding $\{f(D)\}^{-1}$ in a series of powers of D and operating on x^m by each term of the series. Since $D^r x^m = 0$ when $r > m$ the result will be a finite number of terms.

EXAMPLE 1. $(D^2-D+1)y = x^3$.

The complementary function is $e^{\frac{1}{2}x}(A\cos\frac{1}{2}\sqrt{3}x + B\sin\frac{1}{2}\sqrt{3}x)$. A particular integral is

$$\frac{1}{1-D+D^2}x^3 = (1-\overline{D-D^2})^{-1}x^3$$

$$= \{1+D-D^2+(D-D^2)^2+(D-D^2)^3+\dots\}x^3 = x^3+3x^2-6.$$

EXAMPLE 2. $(D^2+4)y = 3x^5$.

EXAMPLE 3. $(D^3-3D^2+2D)y = 12(x^2-2x+4)$.

EXAMPLES ON CHAPTER VI

1. $(D^4+D^2+1)y = 10$.
2. $(D^2+2D+1)y = 17\cos 4x$.
3. $(3D^2+D-14)y = 13e^{2x}$.
4. $(D^3-D^2-D+1)y = x$.
5. $(D^2-6D+10)y = x^3e^{2x}$.
6. $(D^3-2D+1)y = x^4$.
7. $(D^3+3D^2+3D+2)y = e^{-2x}\sin x$.
8. $(D-a)^4y = e^{ax}$.
9. $(D^4+26D^2+25)y = 24\sin 2x\cos 3x$.

10. $(D-a)^2y = x^a e^{ax}$. 11. $(D+2)^2(D+3)y = e^{-2x}$.

12. $(D^4+5D^2+4)y = 12\sin 2x$.

13. $(6D^2+11D-10)y = 19(e^{\frac{2}{3}x}+e^{-\frac{5}{2}x})$

14. $(D^2-2D+1)y = xe^x\sin x$. 15. $(D^2-9)y = 3-9x^2+27x^4$.

16. $(D^2+5)y = \sin x+2\sin^2 x$.

17. Show that if V is a function of x and the operator $f(D)$ has the form of a polynomial,

(i) $D^n(xV) = xD^nV+nD^{n-1}V$,

(ii) $f(D)(xV) = xf(D)V+f'(D)V$,

(iii) $\dfrac{1}{f(D)}(xV) = \left\{x-\dfrac{1}{f(D)}f'(D)\right\}\dfrac{1}{f(D)}V$.

18. $(D^2+4)y = x\cos x$. 19. $(D-1)y = 2x^2\sin x$.

20. $(D^2+1)y = 4x\cos x$. 21. $(D^4+2D^2+1)y = 8\cos x$.

22. $(D^3+2D^2+D)y = 4x^3-6x^2$.

23. $(D^2+4D+8)y = 12e^{-2x}\sin x\sin 3x$.

24. $(4D^2+8D+3)y = e^{-x}(x^2+\sin\frac{1}{2}x)$.

25. $(D^2+D^4)y = 4+12x^2+4\sin^2\frac{1}{2}x$.

26. $(3D^2+36)y = 16\cos^4 x$.

27. The differential equation for simple harmonic motion is

$$\frac{d^2s}{dt^2}+\alpha^2 s = 0.$$

Show that if the particle starts from its equilibrium position with speed v, its displacement t seconds later is $s = \dfrac{v}{\alpha}\sin\alpha t$.

28. The differential equation for harmonic motion with a damping proportional to the velocity is $\dfrac{d^2s}{dt^2}+2\beta\dfrac{ds}{dt}+\alpha^2 s = 0$. Solve the equation, and discuss the character of the motion for different values of α and β.

29. Show that a particle moving in a straight line according to the equation $\dfrac{d^2s}{dt^2}+\alpha^2 s = A\cos\alpha t$ oscillates with a constant period and ever-increasing amplitude.

30. The motion of a particle on a straight line is given by

$$5\frac{d^2s}{dt^2}+2\frac{ds}{dt}+40s = 0,$$

where s is measured in inches and t in seconds. Show that the particle executes oscillations which die down to about $\frac{1}{10}$ of their original amplitude in 15 seconds.

CHAPTER VII

METHODS APPLICABLE TO CERTAIN LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

When the coefficients in a linear equation are not constant no such simple and effective methods for its solution are available as in the case of constant coefficients The first type of equation to be considered in the present chapter transforms by a change of variable into an equation with constant coefficients and may therefore be solved by the methods of the preceding chapter Next follows a discussion of the rare case in which the linear equation is exact Thirdly a description is given with examples, of the important method of obtaining solutions in the form of infinite series

40 The homogeneous linear equation The so-called homogeneous equation has the form

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}x\frac{dy}{dx}+a_ny=Q$$

in which the a s are constant and Q is a function of x We shall change the independent variable x to z where $x=e^z$ or $z=\log x$ Then

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$$

Also
$$\frac{d^2y}{dx^2}=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

and

$$\frac{d^3y}{dx^3}=-\frac{2}{x^3}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+\frac{1}{x^2}\left(\frac{d^3y}{dz^3}-\frac{d^2y}{dz^2}\right)\frac{dz}{dx}=\frac{1}{x^3}\left(\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}\right)$$

If the symbol D be used for d/dz these results may be written

$$x\frac{dy}{dx}=Dy$$

$$x^2\frac{d^2y}{dx^2}=D(D-1)y$$

$$x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y$$

These results are readily extended to higher derivatives. The general formula, which may be established by mathematical induction, is

$$x^r\frac{d^ry}{dx^r} = D(D-1)\dots(D-r+1)y.$$

By these substitutions the given differential equation is transformed into an equation with constant coefficients and the methods of Chapter VI are then available for its solution. In particular it follows that the complementary function is found by solving the auxiliary equation,

$$m(m-1)\dots(m-n+1)+a_1m(m-1)\dots(m-n)+ \\ +\dots+a_{n-1}m+a_n = 0.$$

If m_1, m_2,..., m_n are roots of this equation and are all distinct then the solution of the reduced equation is

$$c_1e^{m_1z}+c_2e^{m_2z}+\dots+c_ne^{m_nz}$$

or, in terms of the original variable x,

$$c_1x^{m_1}+c_2x^{m_2}+\dots+c_nx^{m_n}.$$

If m_r is a double root of the auxiliary equation the corresponding terms in the complementary function are $e^{m_rz}(c_r+c_r'z)$ or $x^{m_r}(c_r+c_r'\log x)$.

EXAMPLE 1. $x^2\dfrac{d^2y}{dx^2}-6x\dfrac{dy}{dx}+6y = 0.$

EXAMPLE 2. $x^2\dfrac{d^2y}{dx^2}+9x\dfrac{dy}{dx}+16y = 0.$

EXAMPLE 3. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+2y = 0.$

41. The particular integral for the homogeneous form. When the homogeneous equation is transformed into an equation with constant coefficients we may, as in Chapter VI, use the symbol D for d/dz. It is possible, however, to avoid explicit reference to the variable z by using instead of D a new symbol θ, so that

$$x\frac{d}{dx} = \theta, \qquad x^2\frac{d^2}{dx^2} = \theta(\theta-1), \qquad x^3\frac{d^3}{dx^3} = \theta(\theta-1)(\theta-2), \quad \text{etc.}$$

The first-order equation $x\frac{dy}{dx}-ay = Q$ has the solution $= x^a \int x^{-a-1}Q\,dx$ from which follows the formula

$$\frac{1}{\theta-a}Q = x^a \int x^{-a-1}Q\,dx$$

If then the general equation expressed in terms of the operator θ has the form $F(\theta)y = Q$ a particular integral may be found by factoring $F(\theta)$ and employing either of the two methods of art 35

Each of the rules which were worked out in Chapter VI for writing particular integrals corresponding to special forms of the right member Q has its counterpart in the case of the homogeneous equation The only case of sufficient importance to deserve mention is the following The equation $f(\theta)y = x^m$ becomes with the independent variable $z = \log x$ $F(D)y = e^{mz}$ A particular integral is therefore $y = \frac{e^{mz}}{F(m)} = \frac{x^m}{F(m)}$ provided $F(m) \neq 0$ Example 1 shows how to deal with the exceptional case in which $F(m) = 0$

EXAMPLE 1 $\quad x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y = x^2$

In terms of the operator θ this equation is

$$\{\theta(\theta-1)-2\theta+2\}y = x^2 \quad \text{or} \quad (\theta-1)(\theta-2)y = x^2$$

When $x = e^z$ this becomes $(D-1)(D-2)y = e^{2z}$ Hence a particular integral is

$$\frac{1}{(D-1)(D-2)}e^{2z} = \left(\frac{1}{D-2}-\frac{1}{D-1}\right)e^{2z} = ze^{2z}-e^{2z} = x^2(\log x-1)$$

The complementary function is $y = c_1x^2+c_2x$ and the complete solution $y = x^2(\log x+C_1)+c_2x$ where $C_1 = c_1-1$

EXAMPLE 2 $\quad x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}+6y = e^x$

EXAMPLE 3 $\quad x^3\frac{d^3y}{dx^3}+6x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+2y = x^2+3x-4$

42 Exact linear equations If from an equation of order $n-1$ containing an arbitrary constant an equation of order n

is derived by differentiating and eliminating the constant, the former equation is called a first integral of the latter. An equation is called exact if it may be derived from its first integral by differentiation and no further operation. Exact equations are of rare occurrence in the applications of the subject. In case they are linear they may be recognized and the first integrals written down by the rule which follows. If then a first integral is exact, a second integral may be obtained in the same way.

The linear equation to be tested is

$$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_{n-1}\frac{dy}{dx}+P_ny=Q. \quad (1)$$

If the left member is the derivative of a differential expression of order $n-1$ the latter must contain the term $P_0\frac{d^{n-1}y}{dx^{n-1}}$. This term gives on differentiation the first term of (1) and also the term $P'_0\frac{d^{n-1}y}{dx^{n-1}}$. To cancel this latter term and to give also the second term of (1) the first integral must contain $(P_1-P'_0)\frac{d^{n-2}y}{dx^{n-2}}$. This term gives on differentiation the additional term

$$(P'_1-P''_0)\frac{d^{n-2}y}{dx^{n-2}}.$$

This term will be cancelled and the third term of (1) provided for by a term $(P_2-P'_1+P''_0)\frac{d^{n-3}y}{dx^{n-3}}$ in the first integral. By this procedure we find that all the terms except the last in the left member of (1) are obtained by differentiating

$$P_0\frac{d^{n-1}y}{dx^{n-1}}+(P_1-P'_0)\frac{d^{n-2}y}{dx^{n-2}}+(P_2-P'_1+P''_0)\frac{d^{n-3}y}{dx^{n-3}}+$$
$$+\ldots+(P_{n-1}-P'_{n-2}+\ldots(-1)^{n-1}P_0^{(n-1)})y.$$

The derivative of this expression includes also a final term $(P'_{n-1}-P''_{n-2}+\ldots(-1)^{n-1}P_0^{(n)})y$. Then (1) will be exact if and only if this final term is identical with P_ny, that is, if

$$P_n-P'_{n-1}+P''_{n-2}-\ldots(-1)^nP_0^{(n)}=0$$

In case this condition is satisfied a first integral of (1) is

$$P_0\frac{d^{n-1}y}{dx^{n-1}}+(P_1-P_0)\frac{d^{n-2}y}{dx^{n-2}}+$$
$$+\cdots+(P_{n-1}-P'_{n-2}+\cdots(-1)^{n-1}P_0^{(n-1)})y=\int Q\,dx+c$$

In writing out a first integral for a given exact equation it is preferable that the method rather than the formula be used The method may in fact be used for other than linear equations though no convenient test for exactness can be given for the general case

For a linear equation which is not exact the question arises of whether it can be made exact by multiplying it by an integrating factor The following discussion for the second order equation might be extended to equations of higher order Let the equation

$$P_0\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=Q \tag{1}$$

be multiplied by a function $\mu(x)$ Then

$$\mu P_0\frac{d^2y}{dx^2}+\mu P_1\frac{dy}{dx}+\mu P_2y=\mu Q$$

is exact if $$\frac{d^2(\mu P_0)}{dx^2}-\frac{d(\mu P_1)}{dx}+\mu P_2=0,$$

or if $$P_0\frac{d^2\mu}{dx^2}+\left(2\frac{dP_0}{dx}-P_1\right)\frac{d\mu}{dx}+\left(\frac{d^2P_0}{dx^2}-\frac{dP_1}{dx}+P_2\right)\mu=0 \tag{2}$$

Any function $\mu(x)$ which satisfies (2) is an integrating factor of (1) As (2) is generally as difficult to solve as (1) this fact is seldom of assistance in the solution of (1) The relation between the two equations is however an interesting one The student should verify that a solution of the reduced equation of (1) is an integrating factor of (2) On account of this reciprocal relation between (2) and the reduced equation of (1) these equations are called adjoint

EXAMPLE 1 $\quad x^3\frac{d^3y}{dx^3}+9x^2\frac{d^2y}{dx^2}+18x\frac{dy}{dx}+6y=\cos x$

The test for exactness is satisfied since $6-18+18-6=0$

A first integral is

$$x^3\frac{d^2y}{dx^2}+(9x^2-3x^2)\frac{dy}{dx}+(18x-12x)y = \sin x+c_1$$

or
$$x^3\frac{d^2y}{dx^2}+6x^2\frac{dy}{dx}+6xy = \sin x+c_1.$$

This equation is also exact, giving the second integral

$$x^3\frac{dy}{dx}+3x^2y = -\cos x+c_1x+c_2$$

which is again exact and gives the final solution

$$x^3y = -\sin x+C_1x^2+c_2x+c_3$$

where $C_1 = \frac{1}{2}c_1$.

EXAMPLE 2. $(x^3+2x^2)\frac{d^2y}{dx^2}+2(5x^2+8x)\frac{dy}{dx}+4(5x+6)y = 0.$

This equation is not exact. Its adjoint is found to be

$$(x^3+2x^2)\frac{d^2\mu}{dx^2}-(4x^2+8x)\frac{d\mu}{dx}+(6x+12)\mu = 0$$

or
$$x^2\frac{d^2\mu}{dx^2}-4x\frac{d\mu}{dx}+6\mu = 0,$$

which is of homogeneous type and has the two independent solutions $\mu = x^2$ and $\mu = x^3$. These are two integrating factors either of which makes the given equation exact.

EXAMPLE 3.

$$x^3(x-4)\frac{d^3y}{dx^3}+12x^2(x-3)\frac{d^2y}{dx^2}+36x(x-2)\frac{dy}{dx}+24(x-1)y = 0.$$

EXAMPLE 4. $x^4\frac{d^2y}{dx^2}+(x^3-x)\frac{dy}{dx}-(x^2-1)y = 0.$

43. Solutions in series. For such differential equations as we have been able to solve the solutions have been expressed by the aid of the so-called elementary functions. The usefulness of such functions as the trigonometric, logarithmic, and exponential functions is due partly to the fact that their properties are well known but also to the fact that we possess tables of their values which make it possible to obtain numerical results

readily in problems whose solutions involve these functions These numerical tables are most easily computed by the aid of infinite series These series, moreover, have other important uses, if we start from them as definitions the properties of the functions may be derived For example,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \quad ,$$

and from this series a table of values of the sine may be computed and the science of elementary trigonometry may be built up

Other functions exist which are not met with in elementary mathematics but which may be represented or defined by infinite series When such a function has been denoted by a symbol a table of its values computed, and a catalogue of its properties derived from the series definition, it takes its place in the family of transcendental functions on a par with $\sin x$, $\sin^{-1}x$ e^x, $\log x$, etc In most cases these new transcendental functions arise as the solutions of differential equations Moreover, an infinite series may be the only practicable way in which to express the solution of a differential equation even though the function represented is not of sufficient importance to deserve a name or a symbol In case we find no combination of elementary functions in finite form to satisfy a differential equation we therefore endeavour to find an infinite series which converges for some interval of values of the independent variable and satisfies the equation As a first illustration of the method we shall find the solution in series of the equation $\frac{d^2y}{dx^2} + y = 0$

Assume that this equation can be satisfied by a series of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 +$$

It is proved in the theory of infinite series that the derivatives of a function represented by a power series within its interval of convergence may be obtained by differentiating the series term by term Consequently

$$\frac{d^2y}{dx^2} = 2a_2 + 2\ 3a_3 x + 3\ 4a_4 x^2 + 4\ 5a_5 x^3 +$$

Hence

$$\frac{d^2y}{dx^2}+y = (2a_2+a_0)+(2.3a_3+a_1)x+(3.4a_4+a_2)x^2+\dots.$$

In order that the equation be satisfied this last series must be identically zero. A power series can vanish identically only when each coefficient vanishes. Hence $a_2 = -\frac{1}{2}a_0$, $a_3 = -\frac{a_1}{3!}$, $a_4 = -\frac{a_2}{3.4} = \frac{a_0}{4!}$, $a_5 = -\frac{a_3}{4.5} = \frac{a_1}{5!}$. Similarly, $a_6 = -\frac{a_0}{6!}$, $a_7 = -\frac{a_1}{7!}$, $a_8 = \frac{a_0}{8!}$, $a_9 = \frac{a_1}{9!}$, etc. Hence the series which satisfies the differential equation is

$$y = a_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right)+a_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right).$$

It is readily established that each series in parentheses converges for all values of x. Since the equation is satisfied whatever be the values of a_0 and a_1, these are arbitrary constants and we have found the general solution of the equation valid for all values of x. This solution, of course, we recognize as $y = a_0\cos x+a_1\sin x$.

Consider next an equation whose solutions are not elementary functions, $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy = 0$ (Bessel's equation of the zeroth order). As before, we assume the series

$$y = a_0+a_1x+a_2x^2+a_3x^3+\dots$$

from which $$\frac{dy}{dx} = a_1+2a_2x+3a_3x^2+4a_4x^3+\dots$$

and $$\frac{d^2y}{dx^2} = 2a_2+2.3a_3x+3.4a_4x^2+\dots.$$

We multiply the first and third series by x, add the three resulting series, and equate to zero the coefficient of each power of x. This gives $a_1 = 0$, $2a_2+2a_2+a_0 = 0$, $2.3a_3+3a_3+a_1 = 0\dots$, and generally

$$n(n+1)a_{n+1}+(n+1)a_{n+1}+a_{n-1} = 0 \quad\text{or}\quad (n+1)^2a_{n+1} = -a_{n-1}$$

from which $a_{n+1} = -\dfrac{a_{n-1}}{(n+1)^2}$ It follows that

$$a_1 = a_3 = a_5 = \quad = 0$$

and that $a_2 = -\dfrac{a_0}{2^2}$, $a_4 = -\dfrac{a_2}{4^2} = \dfrac{a_0}{2^2 \cdot 4^2}$, $a_6 = -\dfrac{a_0}{2^2\ 4^2\ 6^2}$, etc

Hence $y = a_0\left(1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2\ 4^2} - \dfrac{x^6}{2^2\ 4^2\ 6^2} + \quad\right)$ The series in parentheses, which is easily shown to converge for all values of x, is known as a Bessel function of the zeroth order and is denoted by $J_0(x)$ As $y = a_0 J_0(x)$ contains only one arbitrary constant it is not the general solution of the equation Another independent series solution may be obtained in a less simple manner

44 Continuation. The indicial equation. It may happen that a solution of a differential equation cannot be expressed in a series of positive integral powers of x For example, the functions $e^{1/x}$, $\log x$, $\sqrt{x}$, which may occur in solutions, have no expansions of this form Certain differential equations may be satisfied by series involving negative or fractional powers of x A discussion of all cases will not be attempted here We shall confine our attention to equations of the form $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = 0$ in which P and Q are rational functions of x For such equations we shall illustrate by examples a method of wide generality, leaving aside the finding of a second independent solution when only one is furnished by this method The method consists in assuming a series of the form $y = x^r(a_0 + a_1 x + a_2 x^2 + \quad)$ and attempting to determine the index r and the coefficients so as to satisfy the differential equation.

EXAMPLE 1 $\quad (2x + x^3)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0$

Assume $y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \quad$ where $a_0 \neq 0$ Then

$$\frac{dy}{dx} = ra_0 x^{r-1} + (r+1)a_1 x^r + (r+2)a_2 x^{r+1} + \quad ,$$

$$\frac{d^2y}{dx^2} = (r-1)ra_0 x^{r-2} + r(r+1)a_1 x^{r-1} + (r+1)(r+2)a_2 x^r +$$

Substitute these series in the given equation and equate to zero the coefficients of powers of x. The lowest power of x is x^{r-1} and the resulting equation is $\{2(r-1)r+r\}a_0 = 0$. Since $a_0 \neq 0$, $2(r-1)r+r = 0$. From this equation, which is known as the indicial equation, we determine the values of r. In this case $r = 0$ or $\frac{1}{2}$. Proceeding to the coefficient of x^r we have

$$2r(r+1)a_1+(r-1)ra_0+(r+1)a_1-2a_0 = 0,$$

from which
$$a_1 = -\frac{r-2}{2r+1}a_0.$$

Similarly,
$$a_2 = -\frac{r-1}{2r+3}a_1,$$

$$a_3 = -\frac{r}{2r+5}a_2, \quad \text{etc.}$$

Using first the value $r = 0$, we find $a_1 = 2a_0$, $a_2 = \frac{2}{3}a_0$, $a_3 = a_4 = a_5 = \ldots = 0$, and the corresponding solution reduces to a polynomial, $y = a_0(1+2x+\frac{2}{3}x^2) = a_0 u$, say. Take next the value $r = \frac{1}{2}$. Then

$$a_1 = -\frac{-3}{4}a_0, \qquad a_2 = -\frac{-1}{8}a_1 = \frac{-3.-1}{4.8}a_0,$$

$$a_3 = -\frac{1}{12}a_2 = -\frac{-3.-1.1}{4.8.12}a_0.$$

Similarly,

$$a_4 = \frac{-3.-1.1.3}{4.8.12.16}a_0, \qquad a_5 = -\frac{-3.-1.1.3.5}{4.8.12.16.20}a_0, \quad \text{etc.}$$

Hence

$$y = a_0 x^{\frac{1}{2}}\left(1-\frac{-3}{4}x+\frac{-3.-1}{4.8}x^2-\frac{-3.-1.1}{4.8.12}x^3+\ldots\right)$$

$$= a_0 v, \quad \text{say.}$$

It is easily found by the ratio test that this series converges for values of x numerically less than 2. Hence for these values of x we have two independent solutions. In each case the constant a_0 is arbitrary. If we call it a in the first case and b in the second, the general solution is $y = au+bv$.

EXAMPLE 2 $(1-x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+6y=0$ (Legendre's equation of order two)

Proceeding to satisfy this equation by a series we may set down the work as follows

$$
\begin{array}{r|l}
6 & y = a_0x^r+a_1x^{r+1}+a_2x^{r+2}+\cdots \\
-2x & \dfrac{dy}{dx} = ra_0x^{r-1}+(r+1)a_1x^r+(r+2)a_2x^{r+1}+\cdots \\
1-x^2 & \dfrac{d^2y}{dx^2} = (r-1)ra_0x^{r-2}+r(r+1)a_1x^{r-1}+\cdots \\
 & \qquad\qquad +(r+1)(r+2)a_2x^r+\cdots
\end{array}
$$

The indicial equation is $r(r-1)=0$ whence $r=0$ or 1. Next $r(r+1)a_1=0$. Hence if $r=0$, a_1 is arbitrary and if $r=1$, $a_1=0$. Next,

$$(r+1)(r+2)a_2-(r-1)ra_0-2ra_0+6a_0=0$$

whence
$$a_2=\frac{(r-2)(r+3)}{(r+1)(r+2)}a_0$$

Similarly
$$a_3=\frac{(r-1)(r+4)}{(r+2)(r+3)}a_1$$

$$a_4=\frac{r(r+5)}{(r+3)(r+4)}a_2,\ \text{etc}$$

Now if $r=0$, $a_2=-\dfrac{2\cdot3}{1\cdot2}a_0$, $a_4=0=a_6=a_8=\cdots$

$$a_3=\frac{-1\cdot4}{2\cdot3}a_1 \qquad a_5=\frac{-1\cdot1\cdot4\cdot6}{2\cdot4\cdot3\cdot5}a_1$$

$$a_7=\frac{-1\cdot1\cdot3\cdot4\cdot6\cdot8}{2\cdot4\cdot6\cdot3\cdot5\cdot7}a_1 \quad \text{etc}$$

Hence
$$y=a_0(1-3x^2)+a_1\left(x+\frac{-1\cdot4}{2\cdot3}x^3+\frac{-1\cdot1\cdot4\cdot6}{2\cdot4\cdot3\cdot5}x^5+\cdots\right)$$

Again if $r=1$, $a_1=a_3=a_5=\cdots=0$ and $a_2=-\dfrac{1\cdot4}{2\cdot3}a_0$, $a_4=\dfrac{-1\cdot1\cdot4\cdot6}{2\cdot4\cdot3\cdot5}a_0$ etc. Hence

$$y=a_0\left(x+\frac{-1\cdot4}{3!}x^3+\frac{-1\cdot1\cdot4\cdot6}{5!}x^5+\cdots\right)$$

It now appears that this series is already contained in the solution for $r = 0$. Hence for $|x| < 1$, which is the interval of convergence of the series, the general solution of the equation is obtained from one root of the indicial equation.

EXAMPLE 3. Obtain the indicial equation for $\frac{d^2y}{dx^2}+y = 0.$ Show that the general solution is given by one root and that the solution given by the other root merely repeats one series.

EXAMPLE 4. Show that for Bessel's equation of order zero, the roots of the indicial equation are equal and only one solution is furnished by the present method.

EXAMPLE 5. $3x\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y = 0.$

EXAMPLE 6. $2x(1-x)\frac{d^2y}{dx^2}+(1-9x)\frac{dy}{dx}-3y = 0.$

EXAMPLE 7. $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+12y = 0.$

45. Equations of Bessel, Legendre, and Riccati and the hypergeometric equation. In illustrating the method of integration in series we have confined attention to equations of the form $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy = 0$ where P and Q are rational functions of x. Included under this form and solved by this method are three equations of importance in various branches of mathematics. They are

(i) Bessel's equation,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y = 0,$$

(ii) Legendre's equation,

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0$$

or

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y = 0,$$

(iii) the hypergeometric equation,

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{dy}{dx}-\alpha\beta y = 0$$

In the case of (i) and (ii) the character of the solutions depends on the constant n, called the order of the equation (not to be confused with the usual meaning of order of a differential equation) The solutions of (i) with suitable constant factors are called Bessel functions of order n (see also art 77) The indicial equation gives $r = \pm n$ and leads to solutions denoted by $J_n(x)$ and $J_{-n}(x)$ When n is not an integer these solutions are independent and the general solution is $y = AJ_n(x)+BJ_{-n}(x)$ When n is an integer $J_{-n}(x)$ turns out to be a multiple of $J_n(x)$ and then a series solution of another sort $K_n(x)$ † called a Bessel function of the second kind, is necessary for a complete solution

The solutions of (ii) with suitable constant factors are called Legendre functions (see also art 77) If n is an integer, one solution reduces to a polynomial (as in ex 2 art 44) This polynomial is denoted by $P_n(x)$ when the constant factor is chosen so as to satisfy the condition $P_n(1) = 1$ Thus the first four Legendre polynomials are

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \tfrac{3}{2}x^2-\tfrac{1}{2}, \quad P_3(x) = \tfrac{5}{2}x^3-\tfrac{3}{2}x$$

The usefulness of integration in series is not confined to equations of the form considered above An example of another sort is

(iv) Riccati s equation

$$\frac{dy}{dx}+by^2 = cx^m$$

a non linear first-order equation whose solutions must in general be expressed in infinite series. The solution of Riccati s equation may in fact be expressed in terms of Bessel functions

EXAMPLES ON CHAPTER VII

1 $x^3\frac{d^3y}{dx^3}-5x\frac{dy}{dx}+5y = 7$

2 $(x+1)^2\frac{d^2y}{dx^2}-2(x+1)\frac{dy}{dx}-10y = (x+1)^4$

† See B O Peirce *A Short Table of Integrals*, formulae 735 738

3. $(x^4-4x^3)\frac{d^2y}{dx^2}+(8x^3-24x^2)\frac{dy}{dx}+(12x^2-24x)y = 12x^2.$

4. $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-8y = x^2.$

5. $x^2\frac{d^3y}{dx^3}-4x\frac{d^2y}{dx^2}+4\frac{dy}{dx} = 4x^3.$

6. By the method of series find one solution of

$$2x\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-6y = 0.$$

7. Obtain one solution of Bessel's equation of order n.

8. Solve in series Legendre's equation of order one.

9. $2(ax+b)^2\frac{d^2y}{dx^2}+7a(ax+b)\frac{dy}{dx}-3a^2y = ax.$

10. Show that the following equation is exact and solve it:

$$(3x^2y^3-2xy)\frac{d^2y}{dx^2}+(6x^2y-2x)\left(\frac{dy}{dx}\right)^2+(12xy^2-4y)\frac{dy}{dx}+2y^3 = 0.$$

11. $2x^2\frac{d^2y}{dx^2}+15x\frac{dy}{dx}-7y = \sqrt{x}.$

12. Show that Riccati's equation is reduced to a linear equation of the second order by the transformation $y = \frac{1}{bz}\frac{dz}{dx}$.

13. Obtain the logarithmic series as a solution of the differential equation $(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx} = 0.$

14. Obtain the series for $\tan^{-1}x$ as a solution of

$$(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx} = 0.$$

CHAPTER VIII

MISCELLANEOUS METHODS FOR THE SOLUTION OF EQUATIONS OF THE SECOND AND HIGHER ORDERS

In this chapter are collected some methods of solving differential equations of certain special kinds In each case some peculiarity of the equation or some partial knowledge of its solution furnishes a clue to the method employed. It may happen that a particular method is useful only in finding a first integral of an equation leaving the further reduction to some other method In what follows x will be used for the independent and y for the dependent variable unless otherwise stated

46 Equations from which y is missing An equation of order n from which y is missing may be symbolized as

$$F\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots \frac{dy}{dx}, x\right) = 0$$

This is equivalent to an equation of order $n-1$ in the dependent variable $p = dy/dx$ If the solution of this equation can be found in the form $p = f(x)$ then a further integration finds y Again if y and its first $r-1$ derivatives are missing the substitution $d^r y/dx^r = q$ depresses the order of the equation by r

EXAMPLE 1 $\quad x\dfrac{d^2y}{dx^2}+2x\left(\dfrac{dy}{dx}\right)^2-\dfrac{dy}{dx}=0$

On substituting p for $\dfrac{dy}{dx}$ we have $x\dfrac{dp}{dx}+2xp^2-p=0$ whence $\dfrac{x\,dp-p\,dx}{p^2}+2x\,dx=0$ which has the solution $-\dfrac{x}{p}+x^2=c$ or $p=\dfrac{x}{x^2-c}$ This in turn has the solution $y=\frac{1}{2}\log(x^2-c)+c_1$

EXAMPLE 2 $\quad (1+x^2)\dfrac{d^2y}{dx^2}=1+\left(\dfrac{dy}{dx}\right)^2$

EXAMPLE 3 $\quad \left(\dfrac{d^3y}{dx^3}\right)^2+\left(\dfrac{d^2y}{dx^2}\right)^2=1$

EXAMPLE 4. $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=4x.$

47. Equations from which x is missing. In this case the substitution of p for dy/dx also leads to an equation of lower order, but the higher derivatives must be expressed with y as independent variable. Thus $\dfrac{d^2y}{dx^2}=p\dfrac{dp}{dy}$, $\dfrac{d^3y}{dx^3}=p^2\dfrac{d^2p}{dy^2}+p\left(\dfrac{dp}{dy}\right)^2$, etc. The method is chiefly useful in the solution of second-order equations.

EXAMPLE 1. $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=\dfrac{dy}{dx}.$

EXAMPLE 2. $2y\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2=1.$

EXAMPLE 3. $\left(\dfrac{dy}{dx}\right)^3=y\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}+\left(\dfrac{d^2y}{dx^2}\right)^2.$

48. The equation $d^2y/dx^2=f(y)$. This equation, which is the form assumed by certain differential equations of mechanics, comes under the type discussed in the preceding article and may be solved by the method there given. An alternative method is to multiply both members of the equation by $2\,dy/dx$, thus making it exact. Then

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}=2f(y)\frac{dy}{dx}.$$

Integration with respect to x gives the first integral

$$\left(\frac{dy}{dx}\right)^2=2\int f(y)\,dy+c.$$

The solution of this equation offers no difficulty apart from the actual integration.

The equation $d^ny/dx^n=f(d^{n-2}y/dx^{n-2})$ takes the form just discussed on the substitution $d^{n-2}y/dx^{n-2}=v$ which is therefore a first step towards its solution.

EXAMPLE 1. $\dfrac{d^2y}{dx^2}+a^2y=0.$

EXAMPLE 2 $\frac{d^2y}{dx^2} = 3a^2y^2$, with the initial condition $\frac{dy}{dx} = 0$ when $y = 0$

EXAMPLE 3 $2\frac{d^2y}{dx^2} = e^y$, with the condition $\frac{dy}{dx} = 1$ when $y = 0$

EXAMPLE 4 $\frac{d^2y}{dx^2} = 2y^3 - y$, with the condition $\frac{dy}{dx} = 0$ when $y = 0$

49 The linear equation of the second order. General solution found from a known solution of the reduced equation In the solution of the linear equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \tag{1}$$

a method which is sometimes effective is to replace y by the product of two variables, the object then being to take for one of these variables such a function that the resulting equation may be solved for the other one Thus, if $y = uv$, (1) becomes

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R \tag{2}$$

If now any solution of the reduced equation of (1) be known, let this be substituted for u Then $\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$ and (2) becomes

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} = R, \tag{3}$$

which is a linear equation of the first order in the variable $p = dv/dx$ If v is the general solution of (3), then $y = uv$ is the general solution of (1) It is to be noted that in order to use this method the complete complementary function of (1) need not be known but merely any integral belonging to it Such an integral may sometimes be noticed by an inspection of the equation

EXAMPLE 1 $x\frac{d^2y}{dx^2} - (3x+1)\frac{dy}{dx} + (2x+1)y = e^{2x}$

The left member of this equation reduces to zero on the sub

stitution $y = e^x$. (It is clear that this will happen for any linear equation in which the sum of the coefficients of the left member is zero.) Hence substitute $y = e^x v$. The equation becomes

$$\frac{d^2v}{dx^2}-\left(1+\frac{1}{x}\right)\frac{dv}{dx}=\frac{e^x}{x},$$

whence

$$\frac{dv}{dx}=e^x x\left(\int\frac{1}{xe^x}\cdot\frac{e^x}{x}\,dx+c\right)=xe^x\left(-\frac{1}{x}+c\right)=cxe^x-e^x$$

and $v = c(xe^x-e^x)-e^x+c_1$. The solution of the given equation is accordingly $y = e^{2x}(cx-c-1)+c_1 e^x$.

EXAMPLE 2. $x^2\dfrac{d^2y}{dx^2}-(2x^3+3x)\dfrac{dy}{dx}+(2x^2+3)y = x^5.$

($y = x$ is evidently a solution of the reduced equation.)

EXAMPLE 3. $x^2\dfrac{d^2y}{dx^2}-(x^2+6x)\dfrac{dy}{dx}+(3x+12)y = 0$ given that $y = x^3$ is a solution.

EXAMPLE 4. $\dfrac{d^2y}{dx^2}-(2x+3)\dfrac{dy}{dx}+(2x+2)y = xe^{2x}.$

50. Reduction of the linear equation of the second order to normal form. In the preceding article equation (2) was reduced by choosing for u a function to make the coefficient of v zero. Another way of reducing (2) is to choose for u a function to make the coefficient of dv/dx zero. This function will be a solution of $2\dfrac{du}{dx}+Pu = 0$ from which $u = e^{-\frac{1}{2}\int P\,dx}$, only a particular value of the integral being necessary. To substitute this value of u in (2) we have

$$\frac{du}{dx}=-\tfrac{1}{2}Pe^{-\frac{1}{2}\int P\,dx},\qquad \frac{d^2u}{dx^2}=\left(\tfrac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}\right)e^{-\frac{1}{2}\int P\,dx},$$

and (2) becomes

$$\frac{d^2v}{dx^2}+\left(Q-\tfrac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}\right)v = Re^{\frac{1}{2}\int P\,dx}.$$

This form, which may be abbreviated $\dfrac{d^2v}{dx^2}+Iv = S$, is called

the normal form of a linear equation of the second order It sometimes happens that the normal form of an equation may be solved by one of the methods hitherto considered In that case the solution of the original equation is obtained by multiplying by the factor $e^{-\frac{1}{2}\int P\,dx}$

EXAMPLE 1 $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2+5)y=5e^{x-\frac{1}{2}x^2}$

Here $I=Q-\frac{1}{4}P^2-\dfrac{1}{2}\dfrac{dP}{dx}=x^2+5-x^2-1=4$

and $e^{\frac{1}{2}\int P\,dx}=e^{\frac{1}{2}x^2}$ The normal form of the equation is $d^2v/dx^2+4v=5e^{x-\frac{1}{2}x^2}e^{\frac{1}{2}x^2}=5e^x$, from which

$$v=e^x+a\cos 2x+b\sin 2x$$

It follows that $y=ve^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}x^2}(e^x+a\cos 2x+b\sin 2x)$

EXAMPLE 2 $4x\dfrac{d^2y}{dx^2}-4(x+2)\dfrac{dy}{dx}+(4+x)y=0$

EXAMPLE 3 $(x^3-2x^2)\dfrac{d^2y}{dx^2}+2x^2\dfrac{dy}{dx}-12(x-2)y=0$

51 Variation of parameters In case the complementary function of a linear equation is known the following interesting method of finding a particular integral may be used The method which is due to Lagrange, consists in replacing the arbitrary constants of the complementary function by variables and determining these variables so that the given equation is satisfied The solution of example 1 shows the details of the method and it is clear that the method applies to linear equations of any order with constant or variable coefficients

EXAMPLE 1 $\dfrac{d^2y}{dx^2}+a^2y=\sec ax+\csc ax$ (1)

The complementary function is $A\cos ax+B\sin ax$ Take A and B as functions of x to be determined and set

$$y=A\cos ax+B\sin ax$$

Then

$$\frac{dy}{dx}=-aA\sin ax+aB\cos ax+\cos ax\frac{dA}{dx}+\sin ax\frac{dB}{dx}$$

As a first condition for determining A and B we shall place

$$\cos ax\frac{dA}{dx}+\sin ax\frac{dB}{dx}=0. \tag{2}$$

Then
$$\frac{dy}{dx}=-aA\sin ax+aB\cos ax.$$

Again

$$\frac{d^2y}{dx^2}=-a^2A\cos ax-a^2B\sin ax-a\sin ax\frac{dA}{dx}+a\cos ax\frac{dB}{dx}.$$

Substituting these values of y and d^2y/dx^2 in the given equation, we have

$$-a\sin ax\frac{dA}{dx}+a\cos ax\frac{dB}{dx}=\sec ax+\csc ax.$$

But from (2), $\dfrac{dB}{dx}=-\cot ax\dfrac{dA}{dx}$, whence

$$\frac{dA}{dx}\left(-a\sin ax-a\frac{\cos^2 ax}{\sin ax}\right)=\sec ax+\csc ax$$

or $\dfrac{dA}{dx}=-\dfrac{1}{a}(\tan ax+1)$. Hence $A=\dfrac{1}{a}\left(\dfrac{\log\cos ax}{a}-x\right)$. Also $\dfrac{dB}{dx}=\dfrac{1}{a}(1+\cot ax)$, and $B=\dfrac{1}{a}\left(x+\dfrac{\log\sin ax}{a}\right)$. A particular integral of (1) is therefore

$$y=\frac{1}{a^2}(\cos ax\log\cos ax+\sin ax\log\sin ax)-\frac{x}{a}(\cos ax-\sin ax).$$

EXAMPLE 2. $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y=\dfrac{e^x}{1+e^x}$.

EXAMPLE 3. $x^4\dfrac{d^2y}{dx^2}-4x^3\dfrac{dy}{dx}+6x^2y=\dfrac{x^6}{1+x^2}$, given the complementary function ax^2+bx^3.

52. Change of the independent variable. We have seen in arts. 49 and 50 how a change of the dependent variable may be useful in transforming a linear equation of the second order into an integrable form. Sometimes a change of the independent variable may be determined to achieve the same object. Thus we may transform the equation $\dfrac{d^2y}{dx^2}+P\dfrac{dy}{dx}+Qy=R$ by

replacing x by z where z is a function of x to be chosen. Then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

and
$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

and the given equation becomes

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(P\frac{dz}{dx}+\frac{d^2z}{dx^2}\right)+Qy = R$$

It may be that a value of z in terms of x may be chosen so that this equation takes a simple form One possibility is to choose z so that the coefficient of dy/dz shall disappear Then

$$\frac{d^2z}{dx^2}+P\frac{dz}{dx} = 0 \quad \text{or} \quad \frac{dz}{dx} = e^{-\int P\,dx}$$

from which z is found.

EXAMPLE 1 $\quad x^4\dfrac{d^2y}{dx^2}+2x^3\dfrac{dy}{dx}+4y = e^{1/x}$

Changing the variable x to z, we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(\frac{2}{x}\frac{dz}{dx}+\frac{d^2z}{dx^2}\right)+\frac{4}{x^4}y = \frac{e^{1/x}}{x^4}.$$

Choose z so that $\dfrac{2}{x}\dfrac{dz}{dx}+\dfrac{d^2z}{dx^2} = 0$, that is, $\dfrac{dz}{dx} = \dfrac{1}{x^2}$ and $z = -\dfrac{1}{x}$

The equation becomes, after dividing out z^4, $\dfrac{d^2y}{dz^2}+4y = e^{-z}$,

whence $y = A\cos(2z+\alpha)+\dfrac{e^{-z}}{5} = A\cos\left(\alpha-\dfrac{2}{x}\right)+\dfrac{e^{1/x}}{5}$

EXAMPLE 2 $\quad \tan^2x\dfrac{d^2y}{dx^2}+\tan^3x\dfrac{dy}{dx}-2y = \sin^2x$

EXAMPLE 3 $\quad 4x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}-a^2y = x^2$

EXAMPLES ON CHAPTER VIII

1 $\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2 = 1$

2 $(x+1)\dfrac{d^2y}{dx^2}-2(x+3)\dfrac{dy}{dx}+(x+5)y = e^x$

3 $d^2y/dx^2 = \sin 2y$ with the condition $dy/dx = 0$ when $y = 0$

4. $x^3\dfrac{d^2y}{dx^2}-(2x^2+x)\dfrac{dy}{dx}+(2x+1)y = x^2.$

5. $x^2\dfrac{d^2y}{dx^2}+x^3\dfrac{dy}{dx}-2(x^2+1)y = 0$, given that $y = x^2$ is a solution.

6. $(x+a)\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx} = (x+a)^5.$

7. $x^2\dfrac{d^2y}{dx^2}+x^3\dfrac{dy}{dx}+(x^2-2)y = 0$, given that $1/x$ is a solution.

8. $\sin x\left(\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}\right)+(\sin x+\cos x)y = e^x\cos x$, given that $\sin x$ is an integral of the complementary function.

9. $\dfrac{d^2y}{dx^2}-6x\dfrac{dy}{dx}+9x^2y = 0.$

10. $y^2\left(\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}\right)-(1+2y)\left(\dfrac{dy}{dx}\right)^2 = 0.$

11. By changing the independent variable, solve the equations

$$\text{(i)}\quad x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y = 8x^3\sin x^2,$$

$$\text{(ii)}\quad (1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}+4y = 0.$$

12. Show that $x\dfrac{d^2y}{dx^2}+(2-x)\dfrac{dy}{dx}-y$ may be expressed symbolically in the factored form $(D-1)(xD+1)y$ but that the factors may not be written in reverse order. Use this method to solve the equation

$$x\frac{d^2y}{dx^2}+(2-x)\frac{dy}{dx}-y = e^x.$$

13. By factoring the operator, solve

$$\text{(i)}\quad (x+3)\frac{d^2y}{dx^2}-(2x+7)\frac{dy}{dx}+2y = x+2,$$

$$\text{(ii)}\quad (x+1)\frac{d^2y}{dx^2}-(x+3)\frac{dy}{dx}+2y = (x+1)^3.$$

CHAPTER IX

APPLICATIONS OF EQUATIONS OF THE SECOND ORDER

In Chapter V were considered some applications of differential equations of the first order If we allow our equations to contain second derivatives as well as first the range and variety of applications are greatly increased In geometry for example we may express by means of second-order equations properties of a curve involving curvature In mechanics many problems involving motion with known acceleration or the equilibrium of a system of forces may be formulated and solved by means of second-order equations

53 Problems involving the curvature of a curve The curvature of a curve is given in Cartesian coordinates by the formula

$$\kappa = \frac{d^2y}{dx^2} \bigg/ \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}$$

and the radius of curvature is $1/\kappa$ Any relation among quantities which depend on the curvature the slope and the coordinates at an arbitrary point on the curve may be expressed as an equation of the second order For a relatively small number of such equations the solutions may be obtained by the methods of the preceding chapters

Example 1 Among all the curves for which the curvature is proportional to the length of the normal determine those which are everywhere concave towards the x axis and cross it at right angles

The length of the normal is given by the numerical value of $y\sqrt{(1+p^2)}$ Since the curve is to be everywhere concave towards the x axis it follows that if y is positive d^2y/dx^2 is negative and if y is negative d^2y/dx^2 is positive Hence the differential equation is $-\dfrac{d^2y/dx^2}{(1+p^2)^{3/2}} = k\, y\sqrt{(1+p^2)}$ Since x is missing from this equation we write it in the form $-\dfrac{p(dp/dy)}{(1+p^2)^{3/2}} = k\cdot y\sqrt{(1+p^2)}$

or $-\dfrac{p\,dp}{(1+p^2)^2} = k^2y\,dy$, from which $\dfrac{1}{2(1+p^2)} = \dfrac{k^2y^2}{2}+c$. Since the curve is to cross the x-axis at right angles, then, when y approaches zero, p becomes infinite. Hence $c = 0$ and

$$1+p^2 = \frac{1}{k^2y^2} \quad \text{or} \quad p^2 = \frac{1-k^2y^2}{k^2y^2}.$$

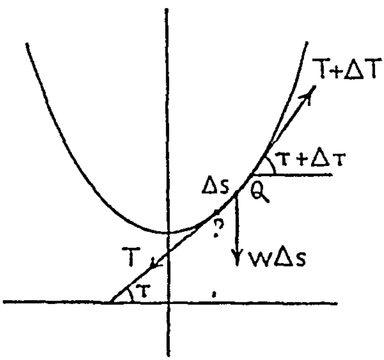
This equation, with variables separated, becomes

$$\frac{ky\,dy}{\sqrt{(1-k^2y^2)}} = dx$$

from which $-\dfrac{\sqrt{(1-k^2y^2)}}{k} = x+c$ or $(x+c)^2+y^2 = 1/k^2$, which denotes a set of circles of radius $1/k$ with centres on the x-axis.

EXAMPLE 2. Determine the equation of the family of circles of given radius from the condition that the radius of curvature is constant.

54. The catenary. If the ends of a uniform flexible chain are fixed at two points, the curve in which the chain hangs is called a catenary. The name is properly applied to an open curve extending upwards indefinitely, and such that if any two points on it be made the points of suspension the lower portion of the curve is the form in which a chain of proper length will hang. Since the curve has clearly a vertical axis of symmetry we take this as the y-axis. The position of the origin will be left undetermined for the present. To find the equation of the curve we must translate into



mathematical language the statement that any portion of the chain is in equilibrium under all the forces which act on it. Acting on a segment PQ are three forces, the tension at P, the tension at Q, and the weight of the segment PQ. Denote the

point P by $(x\ y)$ the tension at P by T, and the inclination of the tangent by τ The corresponding quantities for Q differ from these by increments as shown in the figure If the arc PQ is denoted by Δs, the weight of the segment is $w\Delta s$, w being the weight of a unit length of chain Since the horizontal components of the tensions at P and Q must balance each other, we have $T\cos\tau = (T+\Delta T)\cos(\tau+\Delta\tau)$ Each of these quantities is, in fact equal to the horizontal component of the tension at any point of the chain, which in turn is equal to the tension T_0 at the lowest point That is,

$$T = \frac{T_0}{\cos\tau} \tag{1}$$

Again the increase in the vertical component of the tension from P to Q must balance the weight $w\Delta s$ Hence

$$\Delta(T\sin\tau) = w\Delta s$$

which by use of (1) becomes $\Delta(T_0\tan\tau) = w\Delta s$ Dividing both sides by Δx and letting Δx approach zero we get the differential equation $T_0\dfrac{d}{dx}\tan\tau = w\dfrac{ds}{dx}$ or, in x and y coordinates,

$$T_0\frac{d^2y}{dx^2} = w\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} \tag{2}$$

Set $T_0/w = a$ and denote dy/dx by p Then $a\dfrac{dp}{dx} = \sqrt{(1+p^2)}$ or $a\dfrac{dp}{\sqrt{(1+p^2)}} = dx$ whence $a\log\{p+\sqrt{(1+p^2)}\} = x+c$ At the lowest point on the curve $x = 0$ and $p = 0$ Hence $c = 0$ and $\log\{p+\sqrt{(1+p^2)}\} = x/a$, or $p+\sqrt{(1+p^2)} = e^{x/a}$ By inverting each member of this equation and rationalizing the denominator of the left member we get $\sqrt{(1+p^2)}-p = e^{-x/a}$

Hence $\qquad 2p = e^{x/a}-e^{-x/a}$

or $\qquad \dfrac{dy}{dx} = \frac{1}{2}(e^{x/a}-e^{-x/a})$

and $\qquad y = \frac{1}{2}a(e^{x/a}+e^{-x/a})+c_1$

The position of the origin on the axis of symmetry was left undetermined The equation of the catenary will be simplest if it is chosen so that $c_1 = 0$ If $c_1 = 0$ then when $x = 0$,

$y = a$. We therefore take the origin a distance a below the vertex of the curve. The x-axis is then known as the directrix of the catenary. The equation of the curve is now

$$y = \tfrac{1}{2}a(e^{x/a} + e^{-x/a}) \quad \text{or} \quad y = a \cosh \frac{x}{a}.$$

EXAMPLE 1. Show that the tension at any point (x, y) on the chain is wy, i.e. it is equal to the weight of chain which would hang from that point to the directrix.

EXAMPLE 2. Find the form of the cable of a suspension bridge assuming that the only mass supported is that of the roadbed and that the vertical cables are close enough together that the curve may be taken as smooth.

55. Motion in a line or in a plane under known forces. Suppose a particle of mass m moving in a straight line is acted upon by a force F directed along the line. According to Newton's second law of motion the distance x of the particle from a fixed point of the line satisfies the equation $m\dfrac{d^2x}{dt^2} = F$.

Again, if a particle is moving in a plane under the influence of a force, the force may be resolved into two components F_x and F_y parallel to the coordinate axes; the coordinates of the particle then satisfy the equations $m\dfrac{d^2x}{dt^2} = F_x$, $m\dfrac{d^2y}{dt^2} = F_y$.

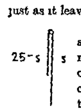
If the moving body is a complex system of particles rather than a single particle, the foregoing equations apply to the motion of its centre of gravity.

EXAMPLE 1. Assuming the Newtonian law of gravitation, find the speed with which a body would strike the earth, falling from the distance of a fixed star.

We assume also the result, proved by Newton from the law of gravitation, that a sphere attracts an outside body as if all the matter in the sphere were concentrated at its centre. Then, if m is the mass of the falling body and r its distance from the earth's centre, we have $m\dfrac{d^2r}{dt^2} = -\dfrac{k}{r^2}$. When $r = R$, the earth's

radius then $d^2r/dt^2 = -g$ Hence $k = mgR^2$ If we* set $dr/dt = -v$ so that v is the speed of the falling body, the equation becomes $mv\,dv = -k\frac{dr}{r^2}$, whence $\frac{1}{2}mv^2 = \frac{k}{r}+c$ When r is the distance to a fixed star $v = 0$, hence c is so small as to be inconsiderable Setting $c = 0$ we have $v = \sqrt{(2k/mr)} = \sqrt{(2gR^2/r)}$ When the body strikes the earth $r = R$ and $v = \sqrt{2gR}$, which in miles per second $= \sqrt{\left(2\frac{32}{5280}3960\right)} = 6\ 9$ (This is called the velocity of escape If a projectile left the earth at right angles to its surface with a velocity greater than this it would not return)

EXAMPLE 2 A chain 25 feet long passes through a small stationary cased in† pulley and hangs at the beginning with 13 feet on one side and 12 feet on the other Neglecting friction, find the time necessary for the chain to slide off and its speed just as it leaves the pulley



Let the weight of 1 foot of the chain be w At any time t measured from the beginning of the motion let s be the length of the longer segment of the chain and let T denote the tension in the chain at the pulley Then, applying the equation of motion to each of the two segments of the chain, we get

$$sw\frac{d^2s}{dt^2} = swg - T,$$

$$(25-s)w\frac{d^2s}{dt^2} = T-(25-s)wg$$

From these equations by addition and dividing out w,

$$25\frac{d^2s}{dt^2} = (2s-25)g$$

or if $ds/dt = v$, $\qquad 25v\,dv = (2s-25)g\,ds$

from which $\qquad 25\frac{v^2}{2} = (s^2-25s)g+c$

† If the pulley is not cased in the centrifugal force will eventually overcome the pressure on the pulley due to the tension and the chain will fly off before the higher end has reached the pulley The chain might, for purposes of the problem, be supposed to pass through an inverted smooth U tube

When $s = 13$, $v = 0$, hence $0 = -156g+c$ or $c = 156g$. Then when $s = 25$, $25\frac{v^2}{2} = 156g$ and $v = \sqrt{\left(\frac{312\times 32}{25}\right)} = 20{\cdot}0$ which, in feet per second, is the speed of the chain as it leaves the pulley. The first integral may now be written

$$\frac{25}{2}\left(\frac{ds}{dt}\right)^2 = (s^2-25s+156)g$$

or

$$\sqrt{\left(\frac{25}{2g}\right)}\frac{ds}{\sqrt{(s^2-25s+156)}} = dt,$$

whence $t+c_1 = \frac{5}{8}\log[s-\frac{25}{2}+\sqrt{\{(s-\frac{25}{2})^2-\frac{1}{4}\}}]$. When $t = 0$, $s = 13$ hence $c_1 = \frac{5}{8}\log\frac{1}{2}$. It follows that

$$t = \tfrac{5}{8}\log[2s-25+\sqrt{\{(2s-25)^2-1\}}].$$

When $s = 25$, $t = \frac{5}{8}\log\{25+\sqrt{(624)}\} = 2{\cdot}44$, which is the number of seconds required for the chain to slide off.

Example 3. If a body falling slowly through a liquid is retarded by a resistance proportional to its velocity, find the distance in terms of the time assuming that the body starts from rest.

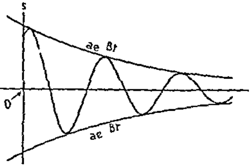
Example 4. A chain 12 feet long rests in a smooth horizontal tube with 1 foot of its length hanging out of the end. Find the time it takes to slide out and its speed as it leaves the tube.

56. Harmonic motion. Resonance. If a particle moves in a straight line with an acceleration which is proportional to the distance of the particle from a fixed point of its path and which is directed towards this fixed point, then the particle is said to execute simple harmonic motion. If s is the distance of the particle from the fixed point, the differential equation of the motion is $\frac{d^2s}{dt^2}+\alpha^2 s = 0$ (see problem 27 et seq., p. 61). The solution of this equation is $s = a\cos(\alpha t-\theta)$, where a and θ are arbitrary constants which in a specific problem may be determined from the initial conditions. When t increases by $2\pi/\alpha$, both s and ds/dt return to their former values, hence the motion is oscillatory with a *period* of $2\pi/\alpha$. As the value of s oscillates between a and $-a$, $2a$ is called the amplitude. This motion is

approximately realized by a vibrating tuning fork or by a simple pendulum swinging through a small angle

If the particle in its harmonic motion encounters a resistance which is proportional to the velocity the acceleration is decreased by an amount which may be denoted by $2\beta\frac{ds}{dt}$ The resulting equation is $\frac{d^2s}{dt^2}+2\beta\frac{ds}{dt}+\alpha^2 s = 0$ The auxiliary algebraic equation is $m^2+2\beta m+\alpha^2 = 0$ which has roots

$$-\beta\pm\sqrt{(\beta^2-\alpha^2)}$$



If $\beta > \alpha$ these roots are real and the differential equation has the solution $s = c_1 e^{\{-\beta+\sqrt{(\beta^2-\alpha^2)}\}t}+c_2 e^{\{-\beta-\sqrt{(\beta^2-\alpha^2)}\}t}$ while if $\beta < \alpha$ the roots are imaginary and the solution of the differential equation is $s = ae^{-\beta t}\cos\{\sqrt{(\alpha^2-\beta^2)}t-\theta\}$ where a and θ are arbitrary constants In the former case each term is of the form ce^{mt} which is a positive decreasing function whose value approaches zero as t increases Hence the displacement s gradually approaches zero without ever passing through that value Such a motion would be realized by a pendulum suspended in some thick liquid in which the resistance is so great that the pendulum sinks to rest without a single oscillation On the other hand if the resistance is small so that $\beta < \alpha$ the second solution applies This represents an oscillation with period $2\pi/\sqrt{(\alpha^2-\beta^2)}$† and with constantly decreasing amplitude

† In this case the term period is to be understood as the interval between two successive zeros of s

It is referred to as damped harmonic motion. This is realized approximately by a pendulum swinging in the air in which case β is small and the oscillations persist for a long time, or by a tuning-fork which is struck and then immersed in water, in which case the vibrations are quickly damped out.

Consider next the case in which the vibrating system is acted upon, in addition to the force of restitution and the damping, by an external force which interferes with the free vibrations. The most interesting case is that in which the applied force is periodic. As an example one may think of a vibrating tuning-fork near one prong of which is placed an electromagnet with an alternating current passing through the coil. In attempting to represent such an applied force it is natural to use a trigonometric function. We shall therefore consider an applied force $k\cos\gamma t$, k and γ being constants. The equation of motion is then

$$\frac{d^2s}{dt^2}+2\beta\frac{ds}{dt}+\alpha^2 s = k\cos\gamma t.$$

The complementary function has already been found (we shall take $\beta < \alpha$). A particular integral is

$$\begin{aligned}\frac{1}{D^2+2\beta D+\alpha^2}k\cos\gamma t &= \frac{D^2+\alpha^2-2\beta D}{(D^2+\alpha^2)^2-4\beta^2D^2}k\cos\gamma t\\ &= k(D^2+\alpha^2-2\beta D)\frac{\cos\gamma t}{(\alpha^2-\gamma^2)^2+4\beta^2\gamma^2}\\ &= k\frac{(\alpha^2-\gamma^2)\cos\gamma t+2\beta\gamma\sin\gamma t}{(\alpha^2-\gamma^2)^2+4\beta^2\gamma^2}\\ &= \frac{k\cos(\gamma t-\phi)}{\sqrt{\{(\alpha^2-\gamma^2)^2+4\beta^2\gamma^2\}}}\end{aligned}$$

where $\tan\phi = \dfrac{2\beta\gamma}{\alpha^2-\gamma^2}$. The complete solution is then

$$s = ae^{-\beta t}\cos\{\sqrt{(\alpha^2-\beta^2)}t-\theta\}+\frac{k}{\sqrt{\{(\alpha^2-\gamma^2)^2+4\beta^2\gamma^2\}}}\cos(\gamma t-\phi).$$

The complementary function gives the free vibrations of the system which in time are damped out, and the particular integral gives the forced vibrations. The amplitude of the forced vibrations is $2k/\sqrt{\{(\alpha^2-\gamma^2)^2+4\beta^2\gamma^2\}}$. If γ be thought of as a variable, the other quantities being fixed, it is easily shown that the

amplitude is greatest when $\gamma = \sqrt{(\alpha^2 - 2\beta^2)}$ If β is small this means that the period of the applied force $2\pi/\gamma$ is nearly equal to that of the free vibrations, $2\pi/\sqrt{(\alpha^2-\beta^2)}$ In this case the forced vibrations which persist after the free vibrations are damped out are of large amplitude

On the other hand if the damping is small and the two periods mentioned nearly equal we may get a better idea of the motion in another way For the case $\beta = 0$ and $\alpha = \gamma$ the equation has a slightly different solution (see ex 29 p 61) The particular integral is $\frac{1}{D^2+\alpha^2} k \cos \alpha t = k \frac{t \sin \alpha t}{2\alpha}$, which shows that the amplitude of the forced vibrations constantly increases with the time Such motion may be approximately realized in various ways Thus for example if soldiers march in step across a bridge and if the period of their step corresponds with one of the natural periods of vibration of the bridge the vibrations may become dangerously large For this reason soldiers break step while crossing a bridge

When the period of the applied force and that of the free vibrations are nearly *in tune* so that the vibrations ultimately reinforce each other the resulting phenomenon is known as resonance

EXAMPLES ON CHAPTER IX

1 Determine the curve which is everywhere concave to the x-axis and for which the radius of curvature is twice the length of the normal

2 Determine the curve which is everywhere convex to the x-axis and for which the radius of curvature is equal to the length of the normal.

3 Determine the curve for which the curvature is at any point equal to the slope of the normal.

4 Determine the curve which at the point (0 2) is tangent to the line $y = 2$ and for which the radius of curvature at any point is equal to one-half the square of the ordinate of that point

5 Determine the curve which touches the x-axis at the origin and for which the length of arc from a fixed point A to a variable point P is proportional to the slope at P

6 A cable supports a weight after the manner of a suspension bridge Assuming that the density of the mass supported is proportional to the horizontal distance from the lowest point of the cable find the curve assumed by the cable

7. Suppose that a set of uniform rods are threaded, like needles, on a string and that each rod hangs vertically. If the string is suspended from its ends and the rods are cut off so that their lower ends are in a horizontal line, find the curve assumed by the string.

8. Solve the following equation of motion, and find the period of the free vibrations, and the period and amplitude of the forced vibrations:

$$2\frac{d^2s}{dt^2} + 2\frac{ds}{dt} + 9s = 8\cos 2t.$$

9. A chain 15 feet long hangs through a small frictionless pulley with 8 feet on one side, of which the end just reaches the floor. In what time will the chain slide off the pulley?

10. If a body falling in air is retarded by a force which varies as the square of the velocity, find the distance and the velocity for a body falling from rest for t seconds.

11. The force of gravity at any point within the earth is that due to the sphere of matter nearer to the centre than the point. If a shaft were sunk to the centre of the earth, find the time necessary for a body falling in it to reach the centre and the velocity with which it would arrive.

12. Set up and solve the equation of motion for a particle which moves in a straight line towards a centre of force if the force varies inversely as the cube of the distance. Assume that the particle starts from rest at a distance a.

CHAPTER X

ORDINARY DIFFERENTIAL EQUATIONS IN MORE THAN TWO VARIABLES

If one independent and two dependent variables are involved the geometrical interpretation of the differential equations requires a space of three dimensions. Moreover it is necessary to consider the case of a single differential equation and that of two simultaneous equations. A preliminary investigation of the geometrical meaning of the equations will show what is to be expected in the way of solutions.

57. Geometric meaning of a pair of simultaneous equations of the first order. If a curve is given by the two equations $u(x, y, z) = 0$ and $v(x, y, z) = 0$, then a tangent line at (x_0, y_0, z_0) on the curve has equations

$$(x-x_0)\left(\frac{\partial u}{\partial x}\right)_0 + (y-y_0)\left(\frac{\partial u}{\partial y}\right)_0 + (z-z_0)\left(\frac{\partial u}{\partial z}\right)_0 = 0$$

$$(x-x_0)\left(\frac{\partial v}{\partial x}\right)_0 + (y-y_0)\left(\frac{\partial v}{\partial y}\right)_0 + (z-z_0)\left(\frac{\partial v}{\partial z}\right)_0 = 0$$

The direction cosines of the tangent are proportional to

$$\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}_0, \qquad \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix}_0, \qquad \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}_0$$

Differentiation of the equations of the curve gives

$$\left.\begin{aligned} \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \\ \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \end{aligned}\right\} \qquad (1)$$

from which it appears that the direction cosines of the tangent are proportional to dx, dy and dz.

Take now a set of two equations of the first order

$$\left.\begin{aligned} P_1 dx + Q_1 dy + R_1 dz = 0 \\ P_2 dx + Q_2 dy + R_2 dz = 0 \end{aligned}\right\} \qquad (2)$$

where coefficients are continuous, single-valued functions of x, y, and z. These equations determine unique values at every point for the ratios $dx : dy : dz$. This may be interpreted as defining at every point a definite direction. We may therefore imagine each point of space provided with a line element, and it is reasonable to expect that these line elements will determine a set of curves of which one passes through every point. Each of these curves will in general cut a given plane and consequently they are as numerous as the points of a plane. In other words, the curves constitute a two-parameter family and require two arbitrary constants in their equations.

58. The equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$. Special devices for solution. When equations (2), art. 57, are solved for the ratios $dx:dy:dz$ we obtain

$$\frac{dx}{Q_1R_2-Q_2R_1}=\frac{dy}{R_1P_2-R_2P_1}=\frac{dz}{P_1Q_2-P_2Q_1}.$$

This form, which we shall abbreviate into

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}, \tag{1}$$

is more convenient for the study of the equations. We saw that the general solution consists of two equations involving two arbitrary constants. We shall sometimes for convenience speak of these equations separately as solutions. Some devices will now be considered which are sometimes useful in obtaining these solutions. (i) If one of the equations in (1) is or can be made free from one of the variables, then it may be possible to solve this equation. Sometimes it is possible in this way to obtain two integral equations, which constitute the general solution. Or, if one solution only is obtained, we may make use of this together with the given equations to get a second differential equation involving two variables. (ii) From (1) we obtain

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{\lambda\, dx+\mu\, dy+\nu\, dz}{\lambda P+\mu Q+\nu R} \tag{2}$$

provided $\lambda P+\mu Q+\nu R\neq 0$, λ, μ, and ν being any multipliers.

Moreover if $\lambda P+\mu Q+\nu R=0$ it is easily shown that

$$\lambda\,dx+\mu\,dy+\nu\,dz=0$$

It is sometimes possible to find multipliers λ μ ν which make $\lambda P+\mu Q+\nu R=0$ and at the same time make

$$\lambda\,dx+\mu\,dy+\nu\,dz=0$$

an integrable equation Again if the fourth ratio in (2) can by proper choice of λ μ ν be made to take the form du/u it may be possible to obtain an integrable equation by making use of the variable u

EXAMPLE 1 $\dfrac{dx}{a}=\dfrac{dy}{y}=\dfrac{dz}{x+z}$

The first equation gives the solution $\dfrac{x}{a}=\log\dfrac{y}{c}$ or $y=ce^{x/a}$

Also $\dfrac{dy}{y}=\dfrac{dx+dz}{x+z+a}$ from which $y=c_1(x+z+a)$ The integral curves are the intersections of the exponential cylinders $y=ce^{x/a}$ with the planes $y=c_1(x+z+a)$

EXAMPLE 2 $\dfrac{dx}{mz-ny}=\dfrac{dy}{nx-lz}=\dfrac{dz}{ly-mx}$

By using successively the multipliers l m n and x y z we find $l\,dx+m\,dy+n\,dz=0$ and $x\,dx+y\,dy+z\,dz=0$ These give the general solution $lx+my+nz=a$ $x^2+y^2+z^2=b$ The curves represented are the intersections of a set of parallel planes with a set of concentric spheres in other words they are circles whose planes are parallel and whose centres lie on a straight line perpendicular to these planes

EXAMPLE 3 $\dfrac{dx}{xy}=\dfrac{dy}{y^2}=\dfrac{dz}{xz}$

EXAMPLE 4 $\dfrac{dx}{y+b}=\dfrac{dy}{x+a}=\dfrac{dz}{z+c}$

EXAMPLE 5 $\dfrac{x\,dx}{z^2}=\dfrac{dy}{y}=\dfrac{z\,dz}{x^2}$

59 Simultaneous linear equations with constant coefficients It is generally an aid towards the solution of a pair of simultaneous equations if one of the dependent variables can

be eliminated. This is possible in a wide variety of cases, the resulting equation being frequently of higher order than either of the given equations. In the case of linear equations in which the coefficients of the dependent variables and their derivatives are constants, the elimination can be effected by an algebraic manipulation of the operator D. An example will show the method.

EXAMPLE 1.
$$\frac{dy}{dx}+\frac{dz}{dx}+2y-z = 3(x^2-e^{-x})$$
$$2\frac{dy}{dx}-\frac{dz}{dx}-y-z = 3(2x-e^{-x}).$$

To eliminate z and dz/dx algebraically would require three equations instead of two. If we differentiate each of the given equations we introduce a second derivative d^2z/dx^2, but we have then four equations from which to eliminate three unknowns, z, dz/dx, and d^2z/dx^2. The two additional equations are

$$\frac{d^2y}{dx^2}+\frac{d^2z}{dx^2}+2\frac{dy}{dx}-\frac{dz}{dx} = 3(2x+e^{-x}),$$
$$2\frac{d^2y}{dx^2}-\frac{d^2z}{dx^2}-\frac{dy}{dx}-\frac{dz}{dx} = 3(2+e^{-x}).$$

The elimination is now effected by adding the first, third, and fourth of these equations and subtracting the second. This gives

$$3\frac{d^2y}{dx^2}+3y = 6e^{-x}+3x^2+6.$$

We note now that the foregoing operations may be condensed as follows. Write the original equations in symbolic form,

$$(D+2)y+(D-1)z = 3x^2-3e^{-x},$$
$$(2D-1)y-(D+1)z = 6x-3e^{-x}.$$

Operate on each term of the first equation by $D+1$ and on each term of the second by $D-1$ and add the results. This gives

$$3(D^2+1)y = 6e^{-x}+3x^2+6$$

which is the equation obtained before. The general solution of

this equation is easily found to be

$$y = e^{-x}+x^2+c_1\cos x+c_2\sin x$$

It remains to find z For this we have a choice of two methods

(1) From the given equations eliminate Dz (by addition in this case) In the result substitute the value just found for y and solve for z Thus

$$(3D+1)y-2z = 3x^2+6x-6e^{-x},$$

whence $$z = 2e^{-x}-x^2+\frac{c_1+3c_2}{2}\cos x-\frac{3c_1-c_2}{2}\sin x$$

(2) Use the same method to find z as was used for y This gives $z = 2e^{-x}-x^2+c_3\cos x+c_4\sin x$ It is to be noted, however, that the constants c_1 c_2 c_3 c_4 are not independent The relations among them may be found by substituting the values found for y and z in either of the given equations and equating coefficients of like terms in the resulting identity Thus, substituting in the first equation we get

$$-e^{-x}+2x-c_1\sin x+c_2\cos x-2e^{-x}-2x-c_3\sin x+$$
$$+c_4\cos x+2e^{-x}+2x^2+2c_1\cos x+2c_2\sin x-$$
$$-2e^{-x}+x^2-c_3\cos x-c_4\sin x = 3x^2-3e^{-x},$$

whence, on equating to zero coefficients of $\cos x$ and of $\sin x$,

$$2c_1+c_2-c_3+c_4 = 0, \qquad -c_1+2c_2-c_3-c_4 = 0,$$

or $$c_3 = \frac{c_1+3c_2}{2}, \qquad c_4 = \frac{c_2-3c_1}{2}$$

EXAMPLE 2 $$\frac{dy}{dx}-z = 0, \qquad \frac{dz}{dx}-y = 0$$

EXAMPLE 3 $$\frac{dy}{dx}+\frac{dz}{dx}-2z = 2\cos x-7\sin x,$$
$$\frac{dy}{dx}-\frac{dz}{dx}+2y = 4\cos x-3\sin x$$

EXAMPLE 4 $$2\frac{dx}{dt}-\frac{dy}{dt}+2x+y = 11t,$$
$$2\frac{dx}{dt}+3\frac{dy}{dt}+5x-3y = 2$$

60. The equation $P\,dx+Q\,dy+R\,dz=0$. Condition for integrability. From the equation $u(x,y,z)=a$, involving an arbitrary constant a, a differential equation

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \tag{1}$$

may be formed. This equation (1) is exact and has $u=a$ as its solution. In order that the equation

$$P\,dx+Q\,dy+R\,dz=0 \tag{2}$$

be exact there must exist a function u such that $\partial u/\partial x=P$, $\partial u/\partial y=Q$, $\partial u/\partial z=R$. From these equations result the following necessary conditions for exactness: $\partial P/\partial y=\partial Q/\partial x$, $\partial Q/\partial z=\partial R/\partial y$, $\partial R/\partial x=\partial P/\partial z$. It is sometimes possible to render (2) exact by means of an integrating factor. In such case (2) is said to be integrable. We proceed now to determine a condition on P, Q, R in order that (2) be integrable. Suppose $\lambda(x,y,z)$ is an integrating factor so that

$$\lambda P\,dx+\lambda Q\,dy+\lambda R\,dz=0$$

is exact. The above conditions for exactness now give

$$\frac{\partial\lambda P}{\partial y}=\frac{\partial\lambda Q}{\partial x},\qquad \frac{\partial\lambda Q}{\partial z}=\frac{\partial\lambda R}{\partial y},\qquad \frac{\partial\lambda R}{\partial x}=\frac{\partial\lambda P}{\partial z}.$$

From the first of these equations $\lambda\frac{\partial P}{\partial y}+P\frac{\partial\lambda}{\partial y}=\lambda\frac{\partial Q}{\partial x}+Q\frac{\partial\lambda}{\partial x}$ or

$$\lambda\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial\lambda}{\partial x}-P\frac{\partial\lambda}{\partial y}.$$

Similarly, $$\lambda\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial\lambda}{\partial y}-Q\frac{\partial\lambda}{\partial z}$$

and $$\lambda\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial\lambda}{\partial z}-R\frac{\partial\lambda}{\partial x}.$$

Multiply these three equations in order by R, P, and Q and add the results. λ is then eliminated and we have the following condition which is necessary for integrability of (2):

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0.$$

This condition is in fact sufficient as well as necessary but the proof will be omitted here

61 Geometric interpretation of $Pdx+Qdy+Rd = 0$ We have seen that geometric meaning may be given to equations in three variables by interpreting dx dy dz as direction numbers of a line element in space As a single equation connecting dx dy dz is not sufficient to determine their ratios it follows that at any point an infinite number of line elements will satisfy

$$P\,dx+Q\,dy+R\,dz=0 \tag{1}$$

The geometric configuration suggested by (1) is therefore a set of curves in space of which an infinite number pass through every point Moreover the equation states that at any specified point a curve whose tangent has direction numbers dx dy dz is perpendicular to a curve whose tangent has direction numbers P Q R A curve of the latter sort is that defined by the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{P} \tag{2}$$

It follows then that through any point the curves which satisfy (1) are at right angles to the single curve which satisfies (2)

Again if (1) is integrable we saw that its solution is of the form $u(x\ y\ z)=a$ In this case the equation may be interpreted as representing either surfaces or curves the latter constituting the family of all curves which lie on the surfaces It follows that in the integrable case of (1) the surfaces which it represents are the orthogonal trajectories of the curves denoted by (2)

If (1) is not integrable there is no difference in its interpretation by means of curves but the interpretation by means of surfaces breaks down The family of curves (2) has in this case no orthogonal surface trajectories

EXAMPLE 1 Interpret geometrically the equation

$$x\,dx+y\,dy+z\,dz=0$$

and consider its relation to $dx/x=dy/y=dz/z$

EXAMPLE 2 The same problem for $x\,dy+y\,dx+z\,dz=0$ in relation to $dx/y=dy/x=dz/z$

EXAMPLE 3. The same for $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$ in relation to $x\,dx=y\,dy=z\,dz$.

62. Solution of $P\,dx+Q\,dy+R\,dz=0$, when integrable. For some integrable equations of this type the solution is evident or may be readily found by inspection. In particular this is the case when the equation is exact or when a proper grouping of the terms makes evident an integrating factor. A method of general application, however, is the following. Let one of the variables, say z, be considered constant and solve the resulting equation in x and y. In this solution replace the arbitrary constant by a function of z and attempt to determine this function so that the given equation is satisfied. Example 2 below illustrates the operation.

Another method may sometimes be used to advantage in case the equation is integrable and the coefficients homogeneous of the same degree. This method, analogous to that used for the homogeneous equation in two variables, consists in the substitution of zu for x and zv for y. In the resulting equation it may happen that z disappears entirely, in which case an equation in the two variables u and v is left. Apart from this exceptional case, however, the substitution leads to an equation in which the variable z is separate from u and v. Since this equation is integrable it is easily shown to be exact.

EXAMPLE 1. $2xz\,dx-2yz^2\,dy-x^2\,dz=0$.

By grouping together the first and third terms we observe that $1/z^2$ is an integrating factor. When it is used the equation takes the form $\frac{2xz\,dx-x^2\,dz}{z^2}-2y\,dy=0$ and the solution is $\frac{x^2}{z}-y^2=c$.

EXAMPLE 2. $2xyz\,dx-z(x^2+z)\,dy-y(x^2+y)\,dz=0$.

Take z constant and hence $dz=0$. In the resulting equation the variables x and y are separable, giving $\frac{2x\,dx}{x^2+z}-\frac{dy}{y}=0$, whence $\frac{x^2+z}{y}=c$. Now replace c by $f(z)$,

$$\frac{x^2+z}{y}=f(z). \qquad (1)$$

Differentiate this and compare the result with the given equation

$$\frac{y(2x\,dx+dz)-(x^2+z)\,dy}{y^2} = f'(z)\,dz$$

or

$$2xy\,dx-(x^2+z)\,dy+\{y-y^2f'(z)\}\,dz = 0$$

When this is multiplied through by z the resulting equation agrees term for term with the given equation provided

$$yz-y^2f'(z) = -y(x^2+y)$$

or

$$yzf'(z) = z+x^2+y = yf(z)+y$$

Hence (1) is a solution provided $zf'(z) = f(z)+1$ or $\dfrac{df(z)}{f(z)+1} = \dfrac{dz}{z}$ from which $f(z)+1 = az$ where a is an arbitrary constant Substituting $f(z) = az-1$ in (1) we obtain the solution

$$\frac{x^2+z}{y} = az-1 \quad\text{or}\quad x^2+z+y = ayz$$

EXAMPLE 3 By setting $x = zu$ $y = zv$ solve

$$(y^2+yz)\,dx+(zx+z^2)\,dy+(y^2-xy)\,dz = 0$$

EXAMPLE 4 $(2xy-3$[illegible]$)\,dx+(x^2+2z^2)\,dy+(4yz-3x)\,dz = 0$

EXAMPLE 5 $z^2y\,dx+(y^2z-z^2x)\,dy-y^3\,dz = 0$

EXAMPLE 6

$$(2xz-yz)\,dx+(2yz-xz)\,dy-(x^2-xy+y^2)\,dz = 0$$

63 Solutions of $P\,dx+Q\,dy+R\,dz = 0$ when not integrable In this case the curves given by $dx/P = dy/Q = dz/R$ have no orthogonal surface trajectories However there is no lack of curves orthogonal to those of this family In fact any arbitrarily given surface is completely covered by curves which satisfy the given equation and which are orthogonal at every intersection to the curves $dx/P = dy/Q = dz/R$ For let $f(x\ y\ z) = a$ denote an arbitrary surface Any curve on it satisfies the equation $\dfrac{\partial f}{\partial x}dx+\dfrac{\partial f}{\partial y}dy+\dfrac{\partial f}{\partial z}dz = 0$ But this forms with the given equation a pair of simultaneous equations which denote a set of curves of which one goes through each point of space Hence the surface $f = a$ is completely covered by these curves

EXAMPLE 1. Find the curves which satisfy the differential equation $(x+z)\,dx+(y+z)\,dy+(z-x-y)\,dz = 0$ and lie on the sphere $x^2+y^2+z^2 = r^2$. The curves must satisfy also the differential equation $x\,dx+y\,dy+z\,dz = 0$. Subtracting this from the given equation we get $z\,dx+z\,dy-(x+y)\,dz = 0$ or

$$\frac{dx+dy}{x+y}-\frac{dz}{z} = 0$$

from which $x+y = kz$ where k is an arbitrary constant. The required curves are the intersections with the sphere of this family of planes.

EXAMPLE 2. Find the curves which satisfy

$$(1+2a)x\,dx+y(1-x)\,dy-z\,dz = 0$$

and lie on the hyperboloid $x^2+y^2-z^2 = c^2$.

EXAMPLE 3. Find the curves which satisfy

$$y\,dx+(y+zx)\,dy-(z+xy)\,dz = 0$$

and lie on the surface $x+y = zx$.

EXAMPLES ON CHAPTER X

1. $\dfrac{x\,dx}{x^2+z^2} = \dfrac{y\,dy}{y^2+z^2} = \dfrac{dz}{2z}$.

2. $\dfrac{dx}{y^2-z^2} = \dfrac{dy}{z^2-x^2} = \dfrac{dz}{x^2-y^2}$.

3. $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{z(x^2-y^2)}$.

4. $\dfrac{dx}{x} = \dfrac{dy}{-2y} = \dfrac{dz}{\cos(x^2y+z)}$.

5. $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{xyz-2x^2}$.

6. $\dfrac{dy}{dx}+2\dfrac{dz}{dx}+y+3z = e^x$, $\quad 2\dfrac{dy}{dx}+5\dfrac{dz}{dx}+6z = x$.

7. $5\dfrac{dy}{dx}+2\dfrac{dz}{dx}+2y+z = \cos x$,

$$7\frac{dy}{dx}+3\frac{dz}{dx}-3y-z = \sin x.$$

8. $\dfrac{d^2x}{dt^2}+\dfrac{dx}{dt}+\dfrac{dy}{dt} = 0$, $\quad \dfrac{dx}{dt}-\dfrac{dy}{dt} = -2e^{-t}\cos t$.

9. $(y-z)\,dx+(z-x)\,dy+(x-y)\,dz = 0$.

10. $y\,dy+z\,dz = \sqrt{(1-y^2-z^2)}\,dx$.

11. $(x\,dx)^2+(y\,dy)^2-(z\,dz)^2+2xy\,dx\,dy = 0$.

12. $yz^2(x^2-yz)\,dx+x^2z(y^2-xz)\,dy+xy^2(z^2-xy)\,dz = 0$.

13. $2xz\,dx+2yz\,dy-(x^2+y^2-z^2)\,dz = 0$.

14 $yz(y+z)\,dx+xz(x+z)\,dy+xy(x+y)\,dz=0$

15 $2x(y^2-z)\,dx+(y^2+z-2x^2y)\,dy+(x^2-y)\,dz=0$

16 $(x^2y-y^3-y^2z)\,dx+(xy^2-x^2z-x^3)\,dy+(xy^2+x^2y)\,dz=0$

17 Find the equations of a curve through the point (1, 1, 1) orthogonal to the family of surfaces $(x^2-y^2-z^2)\,dx+2xy\,dy+2xz\,dz=0$

18 Find the curves on the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ which satisfy

$$a\sqrt{\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)}\,dx+b\sqrt{\left(1-\frac{x^2}{a^2}-\frac{z^2}{c^2}\right)}\,dy+c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\,dz=0$$

19 Find the curves on the cylinders $y^2+z^2=r^2$ (r arbitrary) which satisfy the equation $y^2\,dx-y(2x-1)\,dy+z\,dz=0$

20 In order that the equation $P\,dx+Q\,dy+R\,dz+S\,dt=0$ be integrable it must be integrable when any one variable is constant From this fact state four conditions which are necessary for integrability and show that any one of these may be obtained from the other three The following is a general method of integrating such equations Take z and t constant and solve the resulting equation. Replace the arbitrary constant in the solution by $f(z\ t)$. Differentiate the new equation in the four variables and compare with the original differential equation to find $f(z\ t)$

21 $(y+z+t)\,dx+(z+t+x)\,dy+(t+x+y)\,dz+(x+y+z)\,dt=0$

22 $(y^2+6xzt)\,dx+(2xy-6y^2z^2-t^3)\,dy-(4y^3z-3x^2t-4t^2)\,dz+$
$\quad+(3x^2z-3yt^2+8zt)\,dt=0$

CHAPTER XI

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

PARTIAL differential equations are those in which the derivatives involved are *all* partial derivatives. This implies that at least two variables are independent. This is the case in many physical problems in which, for example, the time and one or more coordinates of an arbitrary point vary independently. As a consequence partial differential equations play a dominant role in mathematical physics. The subject is a large one, and only a brief introduction is in order in an elementary book. The present chapter is concerned with equations in which only first partial derivatives appear. In order to furnish an idea of the kinds of solutions to be expected, we show first how such equations may be derived from their primitives.

64. Formation of partial differential equations by the elimination of arbitrary constants. Let $\phi(x, y, z, a, b) = 0$ be a known relation involving two arbitrary constants a and b and two independent variables which may be taken as x and y. By partial differentiation we obtain

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial y} = 0.$$

From these two and the given equation the constants a and b may be eliminated, the result being an equation of the first order, $f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$ or $f(x, y, z, p, q) = 0$, where we employ the notation $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$.

In the following examples partial differential equations are to be formed corresponding to the given equations, a and b being arbitrary constants to be eliminated.

EXAMPLE 1 $(x-a)^2+(y-b)^2+z^2 = r^2$

By partial differentiation with regard to x and y we get

$$x-a+z\frac{\partial z}{\partial x} = 0, \qquad y-b+z\frac{\partial z}{\partial y} = 0,$$

whence
$$z^2\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1\right\} = r^2$$

or $z^2(p^2+q^2+1) = r^2$

EXAMPLE 2 $(x-a)^2+(y-a)^2+(z-a)^2 = b^2$

EXAMPLE 3 $z = ax+by+ab$

EXAMPLE 4 $z = (x+a)(y+b)$

65 Formation of partial differential equations by the elimination of arbitrary functions. This method has no analogue in the case of ordinary differential equations and is accordingly of peculiar significance in the interpretation of partial differential equations. The equation $z = x+f(y)$ yields by differentiation $\partial z/\partial x = 1$ no matter what function $f(y)$ is Similarly, from $z = f(x+y)$ we deduce $\partial z/\partial x = \partial z/\partial y$ These are particular examples of the following general case Suppose u and v are two known functions of x, y and z and that u and v are connected by an arbitrary functional relation which may be symbolized in the form $\phi(u,v) = 0$ or $v = F(u)$ We assume only that the functions ϕ and F as well as u and v possess first partial derivatives with respect to all their variables Then from $\phi(u,v) = 0$ we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = 0$$

Hence
$$\begin{vmatrix} \dfrac{\partial u}{\partial x}+p\dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial x}+p\dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial y}+q\dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial y}+q\dfrac{\partial v}{\partial z} \end{vmatrix} = 0,$$

which takes the form

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q=\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}$$

or $Pp+Qq = R$ where P, Q, and R, the coefficients in the preceding equation, are known functions of x, y, and z.

The results of the present article suggest that in the general solution of partial differential equations arbitrary functions play the role which is played by arbitrary constants in the case of ordinary equations. This will be borne out in the subsequent discussion.

In the following examples partial differential equations are to be formed by eliminating the arbitrary function f.

EXAMPLE 1. $z = f(xy)$.

EXAMPLE 2. $z = f(ax+by)$.

EXAMPLE 3. $z = f(x^2+y^2)$.

EXAMPLE 4. $f\left(\frac{z}{x},\frac{y}{x}\right) = 0$.

66. The linear equation of the first order. Lagrange's solution. By eliminating the arbitrary function ϕ from the equation $\phi(u, v) = 0$ we obtained an equation of the form

$$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y} = R. \tag{1}$$

This is called the linear equation of the first order.† A relation $u(x, y, z) = a$ will be a solution of (1) provided the derivatives $\partial z/\partial x$ and $\partial z/\partial y$ obtained from it reduce (1) to an identity. From $u = a$ we have

$$\frac{\partial z}{\partial x}= -\frac{\partial u}{\partial x}\bigg/\frac{\partial u}{\partial z}, \qquad \frac{\partial z}{\partial y}= -\frac{\partial u}{\partial y}\bigg/\frac{\partial u}{\partial z}.$$

Hence (1) is satisfied if

$$P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z} = 0, \tag{2}$$

† The distinction should be noted between this definition and that for ordinary differential equations. In this case the equation is linear only in the partial derivatives of the dependent variable, not necessarily in the dependent variable itself.

that is, if the function u satisfies (2) Again, from $u = a$ we obtain

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \tag{3}$$

In order that (2) and (3) be true simultaneously we must have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \tag{4}$$

where the differentials refer to the surface $u = a$ Conversely, if $u = a$ is an integral of (4) then from (4) and (3), (2) follows whence by division by $\partial u/\partial z$ (1) follows Hence $u = a$ will be a solution of (1) if and only if it is an integral of the set (4)

Suppose now that $u = a$ and $v = b$ are two independent integrals of (4) Then $\phi(u, v) = 0$, where ϕ is an arbitrary function is also a solution of (1) For by the elimination of ϕ we obtained (art 65) the equation

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)\frac{\partial z}{\partial x} + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} \tag{5}$$

Also, since $u = a$ and $v = b$ are integrals of (4), we have from (2)

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0,$$

$$P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} + R\frac{\partial v}{\partial z} = 0,$$

whence

$$\frac{P}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}} = \frac{Q}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}} = \frac{R}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}$$

Hence (5) is equivalent to (1)

The foregoing solution of the linear equation is due to Lagrange The ordinary equations (4), upon which the solution depends, are called the subsidiary equations The curves which satisfy (4) and furnish the key to the solution of (1) are called the *characteristic curves* or *characteristics* of (1) For a given function ϕ the solution $\phi(u, v) = 0$ (or $v = F(u)$) represents a surface which is completely covered by characteristics of which one passes through each point on the surface With ϕ arbitrary the solution consists of all surfaces which can be built up from

characteristics. Through an arbitrary curve C which is not itself a characteristic there passes a surface which is the locus of all the characteristics which intersect C. The determination of such a surface through an arbitrary curve C is a particular case of a more general problem first considered by Cauchy.

The foregoing method of solution applies as well when the number of independent variables is greater than two. Thus if $x_1, x_2,..., x_n$ are independent, the form of the linear equation is

$$P_1\frac{\partial z}{\partial x_1}+P_2\frac{\partial z}{\partial x_2}+...+P_n\frac{\partial z}{\partial x_n} = R. \tag{6}$$

The subsidiary equations are

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = ... = \frac{dx_n}{P_n} = \frac{dz}{R},$$

and if $u_i(x, y, z) = a$ $(i = 1, 2,..., n)$ are n independent integrals of this set, the general solution of (6) is $\phi(u_1, u_2,..., u_n) = 0$.

EXAMPLE 1. $a\dfrac{\partial z}{\partial x}+b\dfrac{\partial z}{\partial y} = 1.$

The subsidiary equations are $dx/a = dy/b = dz$. Two independent integrals of this set are $x-az = c_1$ and $y-bz = c_2$. Thus the characteristics are a family of lines parallel to the line $x = az$, $y = bz$. The solution of the given equation may be written $y-bz = F(x-az)$ or $\phi(x-az, y-bz) = 0$ and denotes all surfaces which may be built up from these parallel lines, that is, all cylindrical surfaces with elements parallel to the line $x = az$, $y = bz$.

EXAMPLE 2. $(mz-ny)\dfrac{\partial z}{\partial x}+(nx-lz)\dfrac{\partial z}{\partial y} = ly-mx.$

The subsidiary equations are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}.$$

Two independent integrals of this set were found (art. 58) to be

$$lx+my+nz = a,$$
$$x^2+y^2+z^2 = b.$$

Thus the characteristics are the intersections of a set of concentric spheres with a set of parallel planes. They are therefore

circles which lie in parallel planes, at right angles to the line $x/l = y/m = z/n$, and have their centres on this line The general solution $\phi(lx+my+nz\ x^2+y^2+z^2) = 0$ consists of all surfaces which can be built up from these circles, that is, all surfaces of revolution about the line $x/l = y/m = z/n$.

EXAMPLE 3 $(x-a)\frac{\partial z}{\partial x}+(y-b)\frac{\partial z}{\partial y} = z-c$ Solve and interpret geometrically

EXAMPLE 4 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 0$ Solve and interpret geometrically

67. Standard forms for which solutions are easily found

Under this head we consider four types of equations of the first order and obtain solutions each involving two arbitrary constants

I *Equations with only p and q present* Such an equation, $F(p,q) = 0$, is obviously satisfied by $z = ax+by+c$ provided a and b are chosen to satisfy the relation $F(a,b) = 0$ Thus, for example, $p^2+q+1 = 0$ has the solution $z = ax-(a^2+1)y+c$ An equation may sometimes be reduced to the form $F(p,q) = 0$ by means of a transformation of variables Thus in $f(xp,q) = 0$, if we set $X = \log x$, we have $xp = \partial z/\partial X = P$, say, and the equation becomes $f(P,q) = 0$

II *Equations with only p, q and z present* If in $F(z,p,q) = 0$ we suppose z a function of the single variable $u = x+ay$ where a is an arbitrary constant, we have $\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}$ or $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$ Hence the original equation is satisfied if $F\left(z\ \frac{dz}{du}, a\frac{dz}{du}\right) = 0$ The general solution of this ordinary equation is a solution of the given equation with two arbitrary constants As an example, the ordinary equation corresponding to $p^2+q^2 = 4z$ is $(1+a^2)(dz/du)^2 = 4z$ which gives the solution $(1+a^2)z = (u+b)^2$ or $(1+a^2)z = (x+ay+b)^2$

III *Equations of the form $f(x,p) = \phi(y,q)$* In this equation

the variables are said to be separated. If each side of the equation is set equal to an arbitrary constant a and the resulting equations solved for p and q, the value of z is then found by integrating $dz = p\,dx + q\,dy$. Applying this method to the equation $px^2 = q^2 + y$ we have $p = a/x^2$, $q = \sqrt{(a-y)}$, whence $dz = \frac{a}{x^2}dx + \sqrt{(a-y)}\,dy$ and $z = -\frac{a}{x} - \frac{2}{3}(a-y)^{\frac{3}{2}} + b$.

IV. *Equations of the form* $z = px + qy + f(p, q)$. This equation has a close analogy to Clairaut's equation (art. 17) which is solved by replacing the derivative by an arbitrary constant. This suggests as a trial solution for the present case

$$z = ax + by + f(a, b).$$

From this equation we get by differentiation $p = a$, $q = b$, and hence the differential equation is satisfied.

EXAMPLE 1. $p^2 + q^2 = 4$.

EXAMPLE 2. $pz + q = z^2$.

EXAMPLE 3. $z = px + qy + 4p - q$.

EXAMPLE 4. $yp = xq$.

EXAMPLE 5. $pq = p + q$.

EXAMPLE 6. $z^2 = pq$.

EXAMPLE 7. $\sqrt{(1-x^2)}p = xq(1+y^2)$.

68. Complete, general, and singular integrals.† We have seen that a partial differential equation of the first order may be formed from a primitive by the elimination of either two arbitrary constants or an arbitrary function. In the converse process of solution we have seen how to obtain for the linear equation a primitive involving an arbitrary function and for the equations of art. 67 primitives each involving two arbitrary constants. The latter type of solution, that involving two arbitrary constants, has been named by Lagrange a complete integral, the former involving an arbitrary function, the general integral. It remains to point out that the same equation yields both types of integral and that from an integral of either type

† The discussion of this article applies to partial differential equations with *two* independent variables.

one of the other may be found In the first place it is clear that from the general integral a complete integral may be found by replacing the arbitrary function by a specific function in volving two arbitrary constants In fact any number of com plete integrals may be found in this way and any complete integral appears as a two parameter family of surfaces contained in the general integral The name complete integral appears from this point of view to be a misnomer It gets a measure of justification however from the fact that the general integral may also be obtained from a complete integral This is done in the following manner Let $F(x\ y\ z\ a\ b) = 0$ be a complete integral with the arbitrary constants a and b This denotes a two parameter family of surfaces If we set $b = \phi(a)$ where ϕ is a known function we pick out from the complete integral a one parameter family of surfaces This one parameter family will in general possess a surface envelope whose equation may be found by the usual process At any point of this envelope the values of $x\ y\ z\ p\ q$ determine the equation of the tangent plane and hence agree with the corresponding values for one of the surfaces $F = 0$ † Hence the envelope is itself a solution of the differential equation If now $\phi(a)$ be taken as an arbitrary function we reach the conception of an infinite set of envelopes of one parameter families of the surfaces $F = 0$ These con stitute the general integral If it is possible to find for it a single explicit equation the latter will contain an arbitrary function The elimination process involved in finding an envelope cannot generally be carried out when $\phi(a)$ is arbitrary We may how ever choose particular functions ϕ in a great variety of ways and obtain particular surfaces belonging to the general integral

A third type of integral exists in case all the surfaces of the general integral (which includes any complete integral) possess an envelope This envelope may be found from a complete integral by eliminating a and b from the three equations $F = 0$

† Since $x\ y\ z\ p\ q$ determine a tangent plane we may think of such a set of values as a *plane element* which we may visualize as a small square or circular port on of a plane with the po nt $(x\ y\ z)$ at ts centre Plane elements are useful in the geometric interpretation of part al differential equat ons as are line elements in that of ordinary diffe ential equat ons

$\partial F/\partial a = 0$, $\partial F/\partial b = 0$. In this process it may happen that loci other than envelopes appear. The final test for an envelope is that it satisfy the differential equation.

The three types of integral† are illustrated in the following examples:

EXAMPLE 1. $z = px+qy-k\sqrt{(1+p^2+q^2)}$.

This equation belongs to type IV, art. 67, and has for a complete integral

$$z = ax+by-k\sqrt{(1+a^2+b^2)}.$$

The envelope of this two-parameter family of planes is found by eliminating a and b from this equation and

$$x-k\frac{a}{\sqrt{(1+a^2+b^2)}} = 0,$$

$$y-k\frac{b}{\sqrt{(1+a^2+b^2)}} = 0.$$

This elimination, which is left to the student, gives

$$x^2+y^2+z^2 = k^2.$$

The singular integral is therefore a sphere. The general integral consists of all surfaces which are enveloped by the planes

$$z = ax+\phi(a)y-k\sqrt{[1+a^2+\{\phi(a)\}^2]},$$

where $\phi(a)$ is an arbitrary function. It is clear that any one-parameter family of tangent planes will envelope a developable surface tangent to the sphere and that a surface of this type exists which touches the sphere along any curve on it. All such developable surfaces constitute the general integral which includes, therefore, tangent cylinders, tangent cones, etc. The family of tangent cylinders constitutes a two-parameter family of surfaces which might equally well be taken as a complete integral. A tangent plane would then appear as an envelope of a one-parameter family of these cylinders.

EXAMPLE 2. $z^2(1+p^2+q^2) = r^2$.

This equation was obtained (art. 64, ex. 1) from the primitive $(x-a)^2+(y-b)^2+z^2 = r^2$ which is therefore a complete integral.

† There exist equations of particular forms which possess integrals not included in these three types. Such integrals are termed *special*. Cf. Forsyth, *Differential Equations*, 1914, p. 379.

It consists of all spheres of radius r with centres in the xy plane The singular integral is readily found to be $z^2-r^2=0$ which consists of two planes parallel to the xy plane For the general integral we consider those spheres for which $b=\phi(a)$, that is, we imagine a sphere to move so that its centre traces the curve $y=\phi(x)$ The surface enveloped by the moving sphere is a tubular surface All such surfaces are included in the general integral Again we may pick out from the general integral any number of complete integrals, the set of right circular cylinders of radius r with axes in the xy plane is an example

EXAMPLE 3 $q^2=4z$ Solve and interpret geometrically

69 The method of Lagrange and Charpit. We consider here a general method of obtaining a complete integral of a partial differential equation of the first order,

$$F(x\ y, z\ p\ q)=0 \tag{1}$$

At any point of an integral surface of (1) we have

$$dz=p\,dx+q\,dy \tag{2}$$

If functions p and q in terms of x y, and z are known which satisfy (1) and make (2) an integrable equation, then the solution of (2) is also a solution of (1) Two such functions p and q could satisfy not only (1) but a second independent relation

$$\Phi(x,y,z,p,q)=a \tag{3}$$

The function Φ is subject only to the restriction that (1) and (3) be an integrable system and that $\dfrac{\partial F}{\partial p}\dfrac{\partial \Phi}{\partial q}-\dfrac{\partial F}{\partial q}\dfrac{\partial \Phi}{\partial p}\neq 0$ † Our problem now resolves itself into that of obtaining a function Φ which will answer our purpose To this end we differentiate (1) and (3) partially with regard to x and y In the following equations subscripts indicate derivatives

$$F_x+F_z p+F_p\frac{\partial p}{\partial x}+F_q\frac{\partial q}{\partial x}=0,$$

$$\Phi_x+\Phi_z p+\Phi_p\frac{\partial p}{\partial x}+\Phi_q\frac{\partial q}{\partial x}=0,$$

† This condition means that no relation independent of p and q exists between F and Φ

$$F_y+F_z q+F_p\frac{\partial p}{\partial y}+F_q\frac{\partial q}{\partial y}=0,$$

$$\Phi_y+\Phi_z q+\Phi_p\frac{\partial p}{\partial y}+\Phi_q\frac{\partial q}{\partial y}=0.$$

By elimination of $\partial p/\partial x$ from the first pair and of $\partial q/\partial y$ from the second pair we get

$$F_p\Phi_x-F_x\Phi_p+(F_p\Phi_z-F_z\Phi_p)p+(F_p\Phi_q-F_q\Phi_p)\frac{\partial q}{\partial x}=0,$$

$$F_q\Phi_y-F_y\Phi_q+(F_q\Phi_z-F_z\Phi_q)q+(F_q\Phi_p-F_p\Phi_q)\frac{\partial p}{\partial y}=0.$$

Since $\dfrac{\partial q}{\partial x}=\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{\partial p}{\partial y}$, the last two equations give by addition

$$F_p\Phi_x+F_q\Phi_y+(pF_p+qF_q)\Phi_z-(F_x+pF_z)\Phi_p-(F_y+qF_z)\Phi_q=0.$$

This is a linear equation of the first order with dependent variable Φ and independent variables x, y, z, p, q. By the method of art. 66 a solution is to be found from the system of equations

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{pF_p+qF_q}=\frac{-dp}{F_x+pF_z}=\frac{-dq}{F_y+qF_z}. \tag{4}$$

For our present purposes the general solution of this system is unnecessary. One solution only is needed. The general method may now be summarized as follows. Obtain from the set (4) an integral $\Phi(x,y,z,p,q)=a$ independent of $F=0$. Solve $F=0$ and $\Phi=a$ for p and q in terms of x, y, and z. Substitute these values of p and q in $dz=p\,dx+q\,dy$ and integrate. The resulting solution has two arbitrary constants, a and the constant introduced in the last integration, and is therefore a complete integral.

EXAMPLE. $p^2+q^2=py-qx$.

The set (4) is for this case

$$\frac{dx}{2p-y}=\frac{dy}{2q+x}=\frac{dz}{2p^2+2q^2-py+qx}=\frac{-dp}{q}=\frac{dq}{p}.$$

An integral is obtained from the last two terms, $p^2+q^2=a$. This and the given equation give for p and q the values

$$p=\frac{ay+x\sqrt{\{a(x^2+y^2)-a^2\}}}{x^2+y^2},$$

$$q=\frac{-ax+y\sqrt{\{a(x^2+y^2)-a^2\}}}{x^2+y^2}.$$

Now

$$dz = \frac{ay+x\sqrt{\{a(x^2+y^2)-a^2\}}}{x^2+y^2}dx+\frac{-ax+y\sqrt{\{a(x^2+y^2)-a^2\}}}{x^2+y^2}dy$$

$$= a\frac{y\,dx-x\,dy}{x^2+y^2}+\frac{\sqrt{\{a(x^2+y^2)-a^2\}}}{x^2+y^2}(x\,dx+y\,dy)$$

whence $$z = a\tan^{-1}\frac{x}{y}+\frac{1}{2}\int\frac{\sqrt{(au-a^2)}}{u}\,du,$$

where $u = x^2+y^2$ or

$$z = a\tan^{-1}\frac{x}{y}+\sqrt{(au-a^2)}-a\tan^{-1}\frac{\sqrt{(au-a^2)}}{a}+b$$

EXAMPLES ON CHAPTER XI

NOTE —For the linear equations find the general integrals for the others complete integrals

1 $(y+z)p+(z+x)q = x+y$

2 $xp-yq+x^2-y^2 = 0$

3 $a\dfrac{\partial z}{\partial x_1}+b\dfrac{\partial z}{\partial x_2}+c\dfrac{\partial z}{\partial x_3} = 1$

4 $p^2-pq-pq^2+q^3 = 0$

5 $pq = (x+1)(y-1)$

6 $p^2-pq = 1-z^2$

7 $px\tan y = q$

8 $z^2p^2(p^2-1) = q^2$

9 $(1+x^2)p = (1-y^2)q$

10 $(x^2-y^2-z^2)p+2xyq = 2xz$

11 $px+qy = \sqrt{(x^2+y^2)}$

12 $(2x_1-x_2-x_3)\dfrac{\partial z}{\partial x_1}+(2x_2-x_3-x_1)\dfrac{\partial z}{\partial x_2}+(2x_3-x_1-x_2)\dfrac{\partial z}{\partial x_3} = 0$

13 $z = 2px+2qy+4zp^2+2q$

14 $(x+2z)p+(4xz-y)q = 2x^2+y$

15. $(x^2-yz)p+(y^2-zx)q = z^2-xy$

16 $p^2+q^2-2px-2qy+1 = 0$

17 $2z+p^2+qy+2y^2 = 0$

18 Show that if one of P, Q, R is zero in the equation $Pp+Qq = R$ the characteristics are plane curves in planes parallel to a coordinate plane

19 Find the surfaces whose tangent planes all pass through a fixed point (x_0, y_0, z_0)

20 Find the surface whose tangent plane makes with the coordinate planes in the first octant a tetrahedron of constant volume

21 Find the surface whose tangent plane cuts off from the coordinate axes intercepts whose sum is constant

CHAPTER XII

PARTIAL DIFFERENTIAL EQUATIONS OF ORDER HIGHER THAN THE FIRST

70. General remarks regarding formation and solution of equations. In a partial differential equation of the second order there may occur two first derivatives and three second derivatives. These are often denoted by single letters,

$$p=\frac{\partial z}{\partial x}, \qquad q=\frac{\partial z}{\partial y}, \qquad r=\frac{\partial^2 z}{\partial x^2}, \qquad s=\frac{\partial^2 z}{\partial x \partial y}, \qquad t=\frac{\partial^2 z}{\partial y^2}.$$

Similarly, in equations of higher order there may be four third derivatives, etc. As in the case of first-order equations, partial differential equations of higher order may be formed from primitives by the elimination of arbitrary constants or of arbitrary functions. The general solution of such an equation involves arbitrary functions. Many applications of the subject, however, require solutions satisfying given initial or boundary conditions, and such solutions may often be found without a knowledge of the general solution. In some cases the solution is assisted by an integration with regard to one independent variable. In this case the constant of integration is an arbitrary function of the other independent variable (or variables).

In the first four examples below, f and ϕ are arbitrary functions to be eliminated.

EXAMPLE 1. $y=f(x-at)+\phi(x+at)$.

Partial differentiation gives

$$\frac{\partial y}{\partial x}=f'(x-at)+\phi'(x+at), \qquad \frac{\partial^2 y}{\partial x^2}=f''(x-at)+\phi''(x+at),$$

$$\frac{\partial y}{\partial t}=-af'(x-at)+a\phi'(x+at), \qquad \frac{\partial^2 y}{\partial t^2}=a^2f''(x-at)+a^2\phi''(x+at).$$

Hence
$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}.$$

This equation has a useful interpretation. Let x and y be the coordinates of a moving point and t the time; take a positive. When t increases by an amount A and x by aA, the function

$f(x-at)$ returns to its former value Hence $y = f(x-at)$ represents a curve which as time increases is continually displaced to the right in other words, it represents a wave moving to the right Similarly $y = \phi(x+at)$ represents a wave moving to the left The differential equation represents this double wave motion and is much used in the study of vibrating strings

EXAMPLE 2 $z = f(x)+\phi(y)$

EXAMPLE 3 $z = f(x^2+y^2)+\phi(x^2-y^2)$

EXAMPLE 4 $x+f(z) = z+\phi(y)$

EXAMPLE 5 Solve $\frac{\partial^2 z}{\partial x \partial y} = 2x$

EXAMPLE 6 Solve $\frac{\partial^2 z}{\partial x^2}+a^2 z = 0$ (Treat as an ordinary equation in x and z and make the arbitrary constants functions of y)

71 Particular solutions found by means of exponentials

Exponential functions were found to be of great use in the solution of ordinary linear equations with constant coefficients In the case of the corresponding partial equations these functions are often convenient for finding particular solutions satisfying given boundary conditions A more general method of dealing with the equations of the following examples is given in art 75 and the subsequent articles

EXAMPLE 1 $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = 0$

As a trial solution set $z = e^{lx+my}$ It will satisfy the equation if $l^2+m^2 = 0$ l and m cannot then be both real except in the trivial case when both vanish Suppose we desire a solution satisfying the boundary conditions (1) $z = 0$ when $x = 0$ (2) z approaches 0 when y becomes infinite From (2) m must be real and negative say $m = -p$ Then $l = \pm ip$ and a formal solution is $z = e^{-py}(c_1 e^{ipx}+c_2 e^{-ipx})$ In art 31 we found, in considering the equation $d^2y/dx^2+\beta^2 y = 0$ that the solution $c_1 e^{i\beta x}+c_2 e^{-i\beta x}$ could be replaced by $A\cos\beta x+B\sin\beta x$ Making the same substitution here we have $z = e^{-py}(A\cos px+B\sin px)$

which is easily shown to satisfy the given equation. Since $z = 0$ when $x = 0$, A must be 0. Hence $z = Be^{-p^2y}\sin px$.

EXAMPLE 2. $a^2\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial y}$ with the same boundary conditions as in ex. 1.

EXAMPLE 3. $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$. Find a solution which is finite for all values of x and t and such that $y = 0$ when $x = 0$ and also when $t = 0$.

EXAMPLE 4. $\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 0$. Find a solution which takes the value e^x when $y = x$.

72. Homogeneous linear equations with constant coefficients. The complementary function. Ordinary linear equations with constant coefficients were found to yield to methods of solution in which symbolic operators played a useful role. We shall find symbolic methods of advantage here also and shall denote $\partial/\partial x$ by D, $\partial/\partial y$ by D' so that, for example, $D^2D'z$ stands for $\partial^3 z/\partial^2 x\partial y$. The type of partial differential equation for which symbolic methods are most useful is the so-called homogeneous linear equation with constant coefficients,

$$(D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \dots + a_n D'^n)z = f(x, y). \qquad (1)$$

A part of the argument of art. 28 may be adapted to the present case and the corresponding results will be merely stated. The general solution of (1) consists of any particular integral and the complementary function which is the general solution of

$$(D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \dots + a_n D'^n)z = 0. \qquad (2)$$

We consider first the question of obtaining the complementary function. For the equation $(D - aD')z = 0$ or $p - aq = 0$ Lagrange's solution gives $z = F(y + ax)$. This fact suggests that $F(y + m_1 x)$ will satisfy (2) provided m_1 is a root of the algebraic equation

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0. \qquad (3)$$

This conclusion is easily verified by direct substitution of $F(y + m_1 x)$ in (2). If $m_1, m_2, \dots, m_n$ are the n roots of (3) and

are all distinct the general solution of (2) is

$$z = F_1(y+m_1x)+F_2(y+m_2x)+ \quad +F_n(y+m_nx) \qquad (4)$$

where the F s are arbitrary functions

If (3) has two or more equal roots the solution (4) contains fewer than n independent functions That it is not then the general solution appears from the following example which is a case in point

$$(D^2-2aDD' +a^2 D'^2)z = 0 \quad \text{or} \quad (D-aD')^2z = 0$$

By using $v = (D-aD')z$ this equation becomes $(D-aD')v = 0$ from which $v = F_1(y+ax)$ Hence

$$(D-aD')z = F_1(y+ax) \quad \text{or} \quad p-aq = F_1(y+ax) \qquad (5)$$

The characteristic equations for (5) are

$$dx = -\frac{dy}{a} = \frac{dz}{F_1(y+ax)}$$

One integral is $y+ax = A$ A second integral is got from $\frac{dz}{F_1(A)} = dx$ whence $\frac{z}{F_1(A)}-x = B$ or $\frac{z}{F_1(y+ax)}-x = B$ The general integral of (5) is therefore

$$\frac{z}{F_1(y+ax)}-x = \Phi(y+ax) \quad \text{or} \quad z = xF_1(y+ax)+F_2(y+ax)$$

This result gives the terms in the solution of (2) corresponding to a double root a of (3) This suggests the more general result which may be directly verified that corresponding to an r fold root a of (3) the terms in the solution of (2) are

$$x^{r-1}F_1(y+ax)+x^{r-2}F_2(y+ax)+ \quad +xF_{r-1}(y+ax)+F_r(y+ax)$$

Suppose now that (3) has a pair of conjugate imaginary roots $\alpha\pm i\beta$ The corresponding terms in the solution of (2) are $F_1(y+\alpha x+i\beta x)+F_2(y+\alpha x-i\beta x)$ which are in general imaginary If however we set $F_1 = \phi+i\psi$ and $F_2 = \phi-i\psi$ where ϕ and ψ are arbitrary real functions we get

$$\phi(y+\alpha x+i\beta x)+\phi(y+\alpha x-i\beta x)+$$
$$+i[\psi(y+\alpha x+i\beta x)-\psi(y+\alpha x-i\beta x)]$$

which is real. If, for example, $\phi(u) = u^2$, $\psi(u) = u^{-1}$, the result is

$$(y+\alpha x+i\beta x)^2+(y+\alpha x-i\beta x)^2+i\left(\frac{1}{y+\alpha x+i\beta x}-\frac{1}{y+\alpha x-i\beta x}\right)$$
$$= 2(y+\alpha x)^2-2\beta^2x^2-2\beta x/\{(y+\alpha x)^2+\beta^2x^2\}.$$

EXAMPLE 1. $r = s$.

EXAMPLE 2. $r+6s+9t = 0$.

EXAMPLE 3. $(D^3-2D^2D'-3DD'^2)z = 0$.

EXAMPLE 4. $(D^3-D^2D'+DD'^2-D'^3)z = 0$.

73. Homogeneous linear equations with constant coefficients. The particular integral. The next problem is that of obtaining a particular integral of equation (1), art. 72. Following the notation used in Chapter VI for ordinary equations, we denote this equation by $F(D,D')z = f(x,y)$ and a particular integral by $\dfrac{1}{F(D,D')}f(x,y)$. The operators D and D' obey the following three laws:

(1) they are distributive over the terms of a sum,

(2) they are commutative with constants and with each other,

(3) they obey the index law.

From these facts it follows that the meaning of $F(D,D')f(x,y)$ is unchanged when $F(D,D')$ is resolved into linear factors which may be written in any order. The argument of art. 35 may now be applied to show that the inverse operator $1/f(D,D')$ may be written in factored form or separated into partial fractions.

This brings us to consider the value of $\dfrac{1}{D-aD'}f(x,y)$ which is a solution of the equation

$$(D-aD')z = f(x,y) \quad \text{or} \quad p-aq = f(x,y). \tag{1}$$

The equations of the characteristics of (1) are

$$dx = \frac{dy}{-a} = \frac{dz}{f(x,y)}.$$

One integral is $y+ax = A$. For a second integral we have

$dz = f(x, A-ax)\,dx$ which gives on integration a value of z and hence the formula

$$\frac{1}{D-aD'}f(x,y) = \int f(x, A-ax)\,dx \tag{2}$$

in which, after integration, A is replaced by $y+ax$

EXAMPLE 1 $(D+D')(D-3D')z = ye^x$.

First method $z = \dfrac{1}{(D+D')(D-3D')}ye^x$

Now $\dfrac{1}{D-3D'}ye^x = \int (A-3x)e^x\,dx = (A+3)e^x - 3xe^x$

which, on the substitution of $y+3x$ for A, becomes $(y+3)e^x$. Again

$$\frac{1}{D+D'}(y+3)e^x = \int (A+x+3)e^x\,dx = (A+2)e^x + xe^x$$

which, on the substitution of $y-x$ for A, gives $(y+2)e^x$. The complementary function is $\phi_1(y-x)+\phi_2(y+3x)$ and the general solution, $z = (y+2)e^x + \phi_1(y-x) + \phi_2(y+3x)$

Second method

$$\frac{1}{(D+D')(D-3D')}ye^x = \frac{1}{4D}\left[\frac{1}{D+D'} + \frac{3}{D-3D'}\right]ye^x$$

From (2) we get

$$\frac{1}{D+D'}ye^x = (y-1)e^x \quad \text{and} \quad \frac{1}{D-3D'}ye^x = (y+3)e^x$$

Hence a particular integral is

$$z = \frac{1}{4D}[y-1+3y+9]e^x = \frac{1}{D}(y+2)e^x = (y+2)e^x$$

EXAMPLE 2 $(D^2+4DD'+4D'^2)z = x^2y$

EXAMPLE 3 $(D^2+DD'-6D'^2)z = x\cos y$

EXAMPLE 4 $(D^2-D'^2)z = e^x\sin y$

The analogy with ordinary linear equations would lead us to expect that short methods may sometimes be employed to determine a particular integral. This is the case when $f(x,y)$ has the form $\phi(ax+by)$. We have

$$D\phi(ax+by) = a\phi'(ax+by),$$
$$D'\phi(ax+by) = b\phi'(ax+by)$$

So $D^2\phi = a^2\phi''$, $DD'\phi = ab\phi''$, etc. Hence

$$F(D,D')\phi(ax+by) = F(a,b)\phi^{(n)}(ax+by).$$

If $F(a,b) \neq 0$, this gives by division

$$F(D,D')\frac{\phi(ax+by)}{F(a,b)} = \phi^{(n)}(ax+by)$$

and
$$\frac{1}{F(D,D')}\phi^{(n)}(ax+by) = \frac{\phi(ax+by)}{F(a,b)}. \tag{3}$$

or, if we replace $\phi^{(n)}$ by ϕ,

$$\frac{1}{F(D,D')}\phi(ax+by) = \frac{1}{F(a,b)}\int\int\cdots\int \phi(ax+by)[d(ax+by)]^n.$$

The case in which $F(a,b) = 0$ is an exception to this formula. It is dealt with by means of the following example:

$$(D-aD')z = x^r\phi(y+ax) \quad \text{or} \quad p-aq = x^r\phi(y+ax).$$

A solution of this is found by Lagrange's method to be

$$z = \frac{x^{r+1}}{r+1}\phi(y+ax),$$

whence we get the formula

$$\frac{1}{D-aD'}x^r\phi(y+ax) = \frac{x^{r+1}}{r+1}\phi(y+ax).$$

It follows that

$$\frac{1}{(D-aD')^n}\phi(y+ax)$$
$$= \frac{1}{(D-aD')^{n-1}}x\phi(y+ax) = \ldots = \frac{x^n}{n!}\phi(y+ax). \tag{4}$$

EXAMPLE 5. $(D^3-DD'^2)z = \sin(2x+y)$.

A particular integral is $\dfrac{1}{D^3-DD'^2}\sin(2x+y) = \frac{1}{6}\cos(2x+y)$.

The complementary function is $F_1(y)+F_2(y+x)+F_3(y-x)$.

EXAMPLE 6. $(D-D')^2(D+2D')z = e^{x+y}$.

The complementary function is

$$F_1(y-2x)+F_2(y+x)+xF_3(y+x).$$

A particular integral is

$$\frac{1}{(D-D')^2}\frac{1}{D+2D'}e^{x+y} = \frac{1}{(D-D')^2}\frac{e^{x+y}}{3} = \frac{x^2}{2!}\frac{e^{x+y}}{3} = \frac{1}{6}x^2e^{x+y}.$$

EXAMPLE 7 $(D^3+D^2D'-12DD'^2)z = 5x+y$

EXAMPLE 8 $(D^2-10DD'+25D'^2)z = \log(3x-4y)$

74. Partial differential equations of mathematical physics. Mention will now be made of some of the partial differential equations which have been much studied on account of their use in mathematical physics For the deduction of these equations from physical hypotheses the student is referred to treatises which deal at length with these questions † The interest of the physicist is not so much in obtaining the general solutions as in obtaining particular solutions which satisfy certain boundary conditions For the existence of such solutions the physical problems which give rise to the equations furnish convincing evidence

Elastic vibrations We have already met (art 70) the equation for the transverse vibrations of a stretched string ‡

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \tag{1}$$

Boundary conditions in this problem have to do with the position of the ends of the string and with the shape and velocity of the string at a given instant which is generally the beginning of the motion, the case of a struck piano string which is initially straight and that of a harp string which is plucked and released are examples Similarly, for a stretched membrane which is in equilibrium along the xy plane and is made to vibrate transversely the equation of motion is

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$

Examples of boundary conditions for this case will readily occur to the reader

The flow of heat If heat flows along a conducting rod the temperature at any point depends upon the position of the point

† For example A G Webster, *Partial Differential Equations of Mathematical Physics*

‡ The constant a which figures in this and the following equations depends on physical properties such as the density and tension of the string or membrane, the conductivity of the conductor etc

and upon the time. If x be the abscissa and u the temperature of an arbitrary point of the rod, then u satisfies the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

Boundary conditions here have to do with the position of the ends of the rod and the initial distribution of temperature along it. If the flow of heat is maintained in such a way that the temperature at any point remains constant with the time, we have $\partial u/dt = 0$ and hence $\partial^2 u/\partial x^2 = 0$. For a flow of heat over a flat plate or through a three-dimensional conductor the equations are generalizations of (2). Thus for a three-dimensional flow,

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{3}$$

If the flow is maintained so that the temperature at any point does not change with the time, it is called a steady flow and the temperature then satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{4}$$

which is known as Laplace's equation. If, for a solid conductor bounded by a closed surface, the temperature of the surface be kept constant with respect to the time but continuously variable from point to point it is highly plausible that the temperature at any point within the solid will not vary with the time. In other words, a steady flow of heat will be maintained. The corresponding mathematical problem is that of obtaining a function which throughout a given volume will satisfy Laplace's equation and which will take on a given set of boundary values *on the bounding surface. This is known as Dirichlet's problem.*

Other examples of flow. The same equations as for the flow of heat characterize the flow of electricity, the variable u in this case being the electric potential. Another application of the equations in two and three dimensions is to the so-called irrotational flow of an incompressible fluid, in which u is the velocity potential. This last example is, in fact, useful in providing a visual analogy for any case of flow according to the same equations.

The Newtonian potential In the case of the gravitational attraction of matter a force function or potential function exists with the property that its partial derivatives $\partial u/\partial x$ $\partial u/\partial y$ $\partial u/\partial z$ are the x y and z components of the force at the point $(x\ y\ z)$ In empty space this Newtonian potential function satisfies Laplace s equation Within a solid of continuous density ρ it satisfies Poisson s equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = -4\pi\rho$$

75 The equation $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$ Fourier series We shall now illustrate a method of much use for obtaining particular solutions satisfying boundary conditions Consider the problem of the vibrating string Suppose the string is of length l with its ends fixed that it is originally at rest and distorted into a curve with the equation $y = f(x)$ Then we require a solution of the following equation and boundary conditions

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

$$\text{(i) } y|_{x=0} = 0 \quad \text{(ii) } y|_{x=l} = 0 \quad \text{(iii) } y|_{t=0} = f(x) \quad \text{(iv) } \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$$

We shall seek a solution of the form $y = XT$ where X is a function of x alone and T of t alone The equation will be satisfied if

$$X\frac{d^2T}{dt^2} = a^2T\frac{d^2X}{dx^2},$$

or if

$$\frac{1}{T}\frac{d^2T}{dt^2} = \frac{a^2}{X}\frac{d^2X}{dx^2}$$

Since the left member of this equation is independent of x and the right member independent of t each must be constant If this constant is λ we have

$$\frac{d^2X}{dx^2} = \frac{\lambda}{a^2}X, \qquad \frac{d^2T}{dt^2} = \lambda T \tag{1}$$

Suppose first that $\lambda = 0$ We get $y = XT = (c_1x+c_2)(c_3t+c_4)$ In order that conditions (i) and (ii) be satisfied for all values

of t we must have $c_1 = c_2 = 0$. The solution then becomes the trivial one $y = 0$ which cannot satisfy condition (iii).

Next suppose λ is positive. We get then

$$y = \left(c_1 e^{\frac{\sqrt{\lambda}}{a}x} + c_2 e^{-\frac{\sqrt{\lambda}}{a}x}\right)(c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t}).$$

In order that condition (i) be satisfied

$$(c_1+c_2)(c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t}) = 0$$

for all values of t. Hence $c_2 = -c_1$. Also from (ii)

$$c_1\left(e^{\frac{\sqrt{\lambda}}{a}l} - e^{-\frac{\sqrt{\lambda}}{a}l}\right)(c_3 e^{\sqrt{\lambda}t} + c_4 e^{-\sqrt{\lambda}t}) = 0,$$

whence $e^{\frac{\sqrt{\lambda}}{a}l} - e^{-\frac{\sqrt{\lambda}}{a}l} = 0$ which cannot be true if $\lambda > 0$. This solution therefore cannot satisfy the boundary conditions.

Finally take for λ a negative constant, $-k^2$. Equations (1) become

$$\frac{d^2X}{dx^2} + \frac{k^2}{a^2}X = 0, \qquad \frac{d^2T}{dt^2} + k^2T = 0,$$

whence $y = \left(A\cos\frac{k}{a}x + B\sin\frac{k}{a}x\right)(C\cos kt + D\sin kt)$. May the constants now be chosen to satisfy the boundary conditions? From (i) we get $A = 0$; from (ii) $B\sin kl/a = 0$, and since $B \neq 0$, $kl/a = m\pi$ or $k = ma\pi/l$ where m is an integer. There is now no loss in setting $B = 1$. Then

$$y = \sin\frac{m\pi}{l}x\left(C\cos\frac{ma\pi}{l}t + D\sin\frac{ma\pi}{l}t\right),$$

$$\frac{\partial y}{\partial t} = \sin\frac{m\pi}{l}x\left(-C\frac{ma\pi}{l}\sin\frac{ma\pi}{l}t + D\frac{ma\pi}{l}\cos\frac{ma\pi}{l}t\right).$$

Now (iv) gives $D = 0$. Hence the solution satisfying (i), (ii), and (iv) is $y = C\sin\frac{m\pi x}{l}\cos\frac{ma\pi t}{l}$ for any value of C and any integral value of m. The sum of any finite number of terms of this kind will also be a solution satisfying the same three conditions, as

$$y = \sum_{i=1}^{n} c_i \sin\frac{i\pi}{l}x\cos\frac{ia\pi}{l}t,$$

and it is plausible that the same will be true for an infinite series of such terms provided the requisite conditions of

convergence hold There remains condition (iii) When $t = 0$ the last solution gives

$$\sum_{i=1}^{n} c_i \sin\frac{i\pi}{l}x \tag{2}$$

The question is now raised whether an arbitrary function $f(x)$ can be expressed in a sum of trigonometric terms of this sort, or in an infinite series of such terms This problem was proposed by Fourier and its answer forms an important province of mathematical analysis Briefly it is possible by means of a trigonometric sum (2) to express approximately and by means of an infinite series to express exactly a function $f(x)$ in the interval $0 < x < l$ provided $f(x)$ is subject to certain very mild restrictions Assuming that such a development is possible we may obtain the coefficients of the series as follows Suppose

$$f(x) = c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{2\pi x}{l} + c_3 \sin\frac{3\pi x}{l} + \tag{3}$$

Let us set $\pi x/l = \xi$ then $f(x) = f(l\xi/\pi) = F(\xi)$ and (3) becomes

$$F(\xi) = c_1 \sin\xi + c_2 \sin 2\xi + c_3 \sin 3\xi + \tag{4}$$

Multiply (4) throughout by $\sin n\xi$ and then integrate the two members between the limits 0 and π We assume here that the integral of the infinite series is obtained by integrating term by term Then

$$\int_0^\pi F(\xi)\sin n\xi\, d\xi = c_1 \int_0^\pi \sin\xi \sin n\xi\, d\xi + c_2 \int_0^\pi \sin 2\xi \sin n\xi\, d\xi + \tag{5}$$

Now, if $n \neq m$

$$\int_0^\pi \sin n\xi \sin m\xi\, d\xi = \tfrac{1}{2} \int_0^\pi [\cos(n-m)\xi - \cos(n+m)\xi]\, d\xi$$

$$= \frac{1}{2}\left[\frac{\sin(n-m)\xi}{n-m} - \frac{\sin(n+m)\xi}{n+m}\right]_0^\pi = 0$$

Also $$\int_0^\pi \sin^2 n\xi\, d\xi = \frac{1}{2}\left[\xi - \frac{\sin 2n\xi}{2n}\right]_0^\pi = \tfrac{1}{2}\pi$$

Hence (5) reduces to

$$\int_0^\pi F(\xi)\sin n\xi\, d\xi = c_n \tfrac{1}{2}\pi$$

whence
$$c_n = \frac{2}{\pi}\int_0^\pi F(\xi)\sin n\xi\, d\xi$$
or, if the variable of integration is changed from ξ to x,
$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\, dx.$$
This formula gives the coefficients in the series (3) and with these coefficients the series $y = \sum c_i \sin\frac{i\pi}{l}x\cos\frac{ia\pi}{l}t$ is a solution of the given equation and four boundary conditions.

The series (3) or (4) with the values found for the coefficients is sometimes called a half-range sine series. In like manner a function may be developed into a half-range cosine series. The further development of this subject will be found in books on Fourier series.

76. Laplace's equation in two dimensions. In this article and the next brief discussions of Laplace's equation will be given, using polar coordinates. This is intended to illustrate further the method of solution used in art. 75 and to suggest the connexion between Laplace's equation and certain important ordinary differential equations of the second order. In two dimensions the equation is
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$
For many problems polar coordinates are an advantage, in which case $x = r\cos\theta$, $y = r\sin\theta$, or $r = \sqrt{(x^2+y^2)}$, $\theta = \tan^{-1}y/x$. Then (1) transforms into
$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2}$$
as the student should verify. Suppose we desire a solution $u(r,\theta)$ which is continuous within and on the unit circle $r = 1$ and which takes on a prescribed set of boundary values on this circle,
$$u(1,\theta) = f(\theta), \tag{3}$$
$f(\theta)$ being periodic of period 2π. Applying the method used in

art 75 we set $u = R\Theta$ R being a function of r alone and Θ of θ alone Then

$$\Theta\frac{d^2R}{dr^2}+\frac{\Theta}{r}\frac{dR}{dr}+\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = 0$$

or

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$$

Since the left member is independent of θ and the right independent of r each must be a constant which we may denote by λ Then $d^2\Theta/d\theta^2+\lambda\Theta = 0$ In order that Θ be periodic λ must be positive and in order that it have the period 2π $\sqrt{\lambda}$ must be an integer We shall therefore denote λ by n^2 Then

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-n^2R = 0 \qquad \frac{d^2\Theta}{d\theta^2}+n^2\Theta = 0$$

The equation in R is of the homogeneous type and gives readily $R = Ar^n+Br^{-n}$ Also $\Theta = a\cos n\theta+b\sin n\theta$ Since u is desired to be continuous when $r = 0$ B must be 0 We may take $A = 1$ and hence $u = r^n(a\cos n\theta+b\sin n\theta)$ This is a solution for any zero or positive integral value of n In order to satisfy (3) the question is raised of whether a finite or infinite series

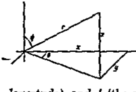
$$\sum(a_n\cos n\theta+b_n\sin n\theta)$$

may be found to represent an arbitrary function $f(\theta)$ This problem again belongs to the subject of Fourier series

77 Laplace's equation in three dimensions In this equation we shall denote the independent variable by V The equation is

$$\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}+\frac{\partial^2V}{\partial z^2} = 0 \tag{1}$$

A homogeneous function of x y and z which satisfies this equation is called a solid spherical harmonic For the study of spherical harmonics it is an advantage to transform the equation to spherical coordinates If these coordinates are r (the distance from the centre) θ (the longitude) and ϕ (the co-latitude) we have



$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\phi$$

In spherical coordinates a solid spherical harmonic has the form $r^n u$, where n is constant and u is a function of θ and ϕ only.

The *transformation of the equation may* be made in two stages by setting $v = r\sin\phi$. For then $x = v\cos\theta$, $y = v\sin\theta$. Hence, regarding z as constant, we have, by (2), art. 76,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial v^2}+\frac{1}{v}\frac{\partial V}{\partial v}+\frac{1}{v^2}\frac{\partial^2 V}{\partial \theta^2}$$

and the left member of (1) becomes, when V is regarded as a function of v, θ, and z,

$$\frac{\partial^2 V}{\partial v^2}+\frac{\partial^2 V}{\partial z^2}+\frac{1}{v}\frac{\partial V}{\partial v}+\frac{1}{v^2}\frac{\partial^2 V}{\partial \theta^2}. \tag{2}$$

It remains to change from the independent variables v and z to r and ϕ. Since $v = r\sin\phi$, $z = r\cos\phi$, we may again use (2), art. 76, regarding θ as constant. Then

$$\frac{\partial^2 V}{\partial v^2}+\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}. \tag{3}$$

Finally the derivative $\partial V/\partial v$ of (2) must be changed to the independent variables r and ϕ. Since $v = r\sin\phi$ and $r = \sqrt{(v^2+z^2)}$, $\phi = \tan^{-1}v/z$, we get

$$\frac{\partial V}{\partial v} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial v}+\frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial v} = \frac{\partial V}{\partial r}\frac{v}{\sqrt{(v^2+z^2)}}+\frac{\partial V}{\partial \phi}\frac{1/z}{1+v^2/z^2}$$

$$= \frac{\partial V}{\partial r}\sin\phi+\frac{\partial V}{\partial \phi}\frac{\cos\phi}{r}. \tag{4}$$

Substituting (3) and (4) in (2) we get Laplace's equation in spherical coordinates,

$$\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\cot\phi}{r^2}\frac{\partial V}{\partial \phi}+\frac{\csc^2\phi}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0. \tag{5}$$

If $V = r^n u$ is a solid spherical harmonic, then u must satisfy the equation

$$n(n-1)r^{n-2}u+\frac{2}{r}nr^{n-1}u+\frac{r^n}{r^2}\frac{\partial^2 u}{\partial \phi^2}+$$
$$+r^n\frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi}+r^n\frac{\csc^2\phi}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

or
$$\frac{\partial^2 u}{\partial \phi^2}+\cot\phi\frac{\partial u}{\partial \phi}+\csc^2\phi\frac{\partial^2 u}{\partial \theta^2}+n(n+1)u = 0. \tag{6}$$

A solution of (6) is called a surface spherical harmonic If u is a function of ϕ alone (6) becomes

$$\frac{d^2u}{d\phi^2}+\cot\phi\frac{du}{d\phi}+n(n+1)u = 0$$

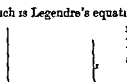
This equation transforms into Legendre's equation by means of the substitution $t = \cos\phi$ For

$$\frac{du}{d\phi} = -\sin\phi\frac{du}{dt}, \qquad \frac{d^2u}{d\phi^2} = -\cos\phi\frac{du}{dt}+\sin^2\phi\frac{d^2u}{dt^2},$$

and hence
$$\sin^2\phi\frac{d^2u}{dt^2}-2\cos\phi\frac{du}{dt}+n(n+1)u = 0$$

or
$$(1-t^2)\frac{d^2u}{dt^2}-2t\frac{du}{dt}+n(n+1)u = 0,$$

which is Legendre's equation (art 45) On account of this connexion with Laplace's equation the Legendre functions are also known as surface zonal harmonics



The transformation of Laplace's equation into cylindrical coordinates is also useful That is, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ By applying (2), art 76, we get the transformed equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}+\frac{\partial^2 V}{\partial z^2} = 0$$

Let us attempt to find a solution of the form $V = R\Theta Z$, these letters denoting functions of r, θ, z respectively By substituting and dividing by $R\Theta Z$ we get

$$\frac{1}{R}\frac{d^2R}{dr^2}+\frac{1}{rR}\frac{dR}{dr}+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

As the first three terms do not depend on z, while the last term is a function of z alone this term must be constant Setting $\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2$ we get $Z = c_1e^{kz}+c_2e^{-kz}$ and also

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+k^2r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$$

By the same argument as before, each member of this equation

is constant. We shall set $-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = m^2$ whence

$$\Theta = c_3 \cos m\theta + c_4 \sin m\theta$$

and
$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(k^2r^2-m^2)R = 0.$$

Changing the independent variable to t where $kr = t$, we have

$$\frac{dR}{dr} = k\frac{dR}{dt}, \qquad \frac{d^2R}{dr^2} = k^2\frac{d^2R}{dt^2},$$

whence
$$t^2\frac{d^2R}{dt^2}+t\frac{dR}{dt}+(t^2-m^2)R = 0.$$

This is Bessel's equation (art. 45). Thus the study of Laplace's equation makes contact at this point with the theory of Bessel functions which from this circumstance are also known as cylindrical harmonics.

EXAMPLES ON CHAPTER XII

Find solutions in the form of exponential functions for 1, 2, and 3.

1. $r+s = p$.
2. $r+s+p-q = 0$.
3. $r-t-2p+2q = 0$.

Find the general solutions of 4, 5, 6, 7.

4. $r-x = t-y$.
5. $6r+11s-7t = 2x-y$.
6. $r-s-6t = xy$.
7. $(D^3-3D^2D'+3DD'^2-D'^3)z = \sin(x-y)$.
8. Verify that the following functions are solid spherical harmonics:

$$ax+by+cz;\ (x^2+y^2+z^2)^{-\frac{1}{2}};\ \tan y/x;\ x^2-y^2;\ 2z^2-x^2-y^2.$$

9. Find a solution of $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 0$ in the form $V = XYZ$, these letters denoting functions of x, y, and z respectively.

10. Represent by a Fourier sine series the function x in the interval from 0 to π.

11. Represent by a Fourier cosine series the function x in the interval from 0 to π.

HISTORICAL NOTE†

In the course of the foregoing treatment the names of various mathematicians have occurred in connexion with particular equations or methods of solution A brief outline of the historical setting of the subject will now be given

Differential equations made their appearance as soon as the calculus had been invented in the latter half of the seventeenth century The formal methods of integration which have been dealt with in this book were nearly all developed during the next hundred years The two founders of the calculus were Sir Isaac Newton (1642—1727) and Gottfried Wilhelm Leibnitz (1646–1716) both of whom contributed to the study of differential equations The following quotation from Ball s *History of Mathematics* shows the trend of Newton s ideas The object of the second or inverse method of fluxions‡ is from the fluxion or some relations involving it to determine the fluent or more generally an equation being proposed exhibiting the relation of the fluxions of quantities to find the relations of those quantities or fluents to one another This is equivalent either to integration which Newton termed the method of quadrature or to the solution of a differential equation which was called by Newton the inverse method of tangents Newton classified some simple types of equations of the first order and made use of power series in obtaining solutions The contributions of Leibnitz included the solution of the equation $y\dfrac{dx}{dy} = X(x)Y(y)$ by quadratures which amounts to a separation of its variables the solution of the homogeneous equation of the first order and of the linear equation of the first order

Contemporary with Newton and Leibnitz and following them was a group of mathematicians who used the new methods of

† For fuller accounts of the history of the subject the reader is referred to W W R Ball, *A Short History of Mathematics* London F Cajori *History of Mathematics* New York or E L Ince *Ordinary Differential Equations* London, 1927 p 529

‡ In Newton s nomenclature the terms fluent and fluxion were used for a function and its derivative or differential coefficient

analysis with great power. Differential equations were studied by these men on account of their intrinsic interest and also because of their occurrence in problems of geometry and mechanics. Foremost among this group were the Swiss mathematicians, James Bernoulli (1654–1705) and his brother John (1667–1748). Among the problems solved by James Bernoulli, who was the first to use the term *integral*, are those of the form of a catenary, the form of an elastic rod fixed at one end and acted on by a force at the other, the form of a sail filled with wind, the curve in which a pendulum should swing in order that its period should be independent of the amplitude, etc. The equation $\frac{dy}{dx}+Py = Qy^n$ was proposed by James Bernoulli and solved by his brother John and by Leibnitz: John Bernoulli made explicit the method of separation of variables; he solved the problem of orthogonal trajectories and the problem of the brachistochrone, i.e. the curve along which a particle falling under the influence of gravity will travel from a point A to a lower point B in the shortest possible time.

The Italian, Count Jacopo Riccati (1676–1754), was first to point out that a second-order equation which does not contain the independent variable explicitly may be reduced to a first-order equation. The equation which bears his name is

$$\frac{dy}{dx} = A(x)y^2+B(x)y+C(x),$$

of which a special case is $\frac{dy}{dx}+ay^2 = bx^m$. Daniel Bernoulli (1700–82), the son of John mentioned above, showed that this equation is integrable in finite form only if m is of the form $-4k/(2k\pm1)$ where k is zero or a positive integer. For other values of m, Riccati's equation is the simplest example of one which cannot be solved by quadratures.

The device of using differentiation to assist in the solution of a differential equation is due to Jean le-Rond d'Alembert (1717–83). The equation $x\phi(y')+y\psi(y')+\chi(y') = 0$ is known as d'Alembert's equation. To the same period belongs Alexis Claude Clairaut (1713–65). The equation known by his name

is a particular case of d Alembert s In its study Clairaut obtained singular solutions† and showed that they were not included in the general solutions

One of the most powerful of the eighteenth century mathematicians in the use of calculus was Leonhard Euler (1707–83) Among his contributions to differential equations was an explicit discussion of integrating factors of which isolated examples had occurred before Euler set up types of equations which can be solved by integrating factors of given forms In his work he made frequent use of the change of variables Much of the theory of linear differential equations is due to Euler He also extended and improved Newton s method of integration in series

The period of development of the elementary methods with which this book deals may be said practically to have closed with the work of Euler By the middle of the eighteenth century the formal methods of solution were known and the attention of mathematicians turned to more fundamental questions One result of the labours of the early analysts was to show that the method of quadratures is insufficient for the solution of other than special cases A more modern point of view was initiated by Joseph Louis Lagrange (1736–1813) He introduced the adjoint equation which is satisfied by any integrating factor of a given equation. He also developed the method of variation of parameters Problems concerning the existence and nature of functions satisfying differential equations of various forms were investigated in the nineteenth century notably by Augustin Louis Cauchy (1789–1857) From the time of Cauchy on the literature of differential equations becomes increasingly voluminous

The subject of partial differential equations presented more difficulties than that of ordinary equations and progress was slower The impetus for their study came first from physical problems The problem of the vibrating string was considered

† The geometrical significance of singular solutions was pointed out by Lagrange but the further analysis of the subject was left to the latter half of the nineteenth century The treatment now used follows the work of Arthur Cayley (1821 95)

early by Brook Taylor (1685–1731) and later by Daniel Bernoulli and Euler. A more significant contribution was made by d'Alembert who gave the general solution of this equation and who encountered partial differential equations in problems on the motion of fluids and the theory of winds. The work of Euler and d'Alembert in this connexion links up with that of Jean Baptiste Joseph Fourier (1768–1830) who gave the equation for the flow of heat. The first successful attempt to deal with the subject systematically was that of Lagrange to whom is due the solution of the linear equation of the first order and the further method known by his name and that of Charpit. Pierre Simon Laplace (1749–1827), in connexion with the idea of potential, introduced the equation known by his name,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

and also the spherical harmonics which are among the solutions of this equation.

The further development of the subject, though of great interest and importance, goes beyond the subject-matter of this first course.

ANSWERS TO EXAMPLES

CHAPTER I

Art. 3 page 5 Ex 2 (i) $dy/dx = m$ (ii) $y = x(dy/dx)+b$
(iii) $y^2(p^2+1) = r^2$ (iv) $(x-y)^2(p^2+1) = r^2(1+p)^2$ where $p = dy/dx$

Art. 4 page 7 Ex 2 $(y-b)\frac{d^2y}{dx^2}+1+\left(\frac{dy}{dx}\right)^2 = 0$

Ex 3 $3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3}$

Examples at end Page 8

1 $\frac{d^3y}{dx^3} = 0$ 2 $\left(y-x\frac{dy}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$ 3. $\frac{dr}{d\theta} = r$

4. $y\,dx - 2x\,dy = 0$ 5 $(x-y)^2(1+p^2) = (x+yp)^2$

6 $(1-p)(xp-y) = kp$ 7 $(px-y)^2+kp = 0$

8 $y\,dy = 2a\,dx$ 9 $xy(xp-y)^2+k^2p = 0$

10 $y(8x-1)dy+(3x^2-3y^2-x+3)dx = 0$

11 $(x+yp)(xp-y) = kp$ 12 $(p^2+1)(x^2+y^2)^2 = 4(px-y)^2$

13 $(1+2p)(1+p^2+yq)-2q(x+yp) = 0$ where $q = d^2y/dx^2$

14 $xyq+xp^2-yp = 0$

CHAPTER II

Art 6 page 10 Ex 2 $x^4-2x^3y+4x^2y^3+y^4 = c$
Ex 3 $e^x+xe^y+y\sin x+\cos y = c$

Art 7 page 11 Ex 2 $x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)} = c$
Fx 3 $xe^x(1+y^2) = c$ Ex 4 $(x-2)(y+3)^3(y-1)^6 = c(x+2)$

Art. 8 page 11 Ex 2 $\log\sqrt{(x^2+y^2)} = \tan^{-1}(y/x)+c$.
Ex 3 $x^2(x^2+2y^2) = c$ Ex. 4. $x^3+y^3 = cx^2$

Art 9 page 12 Fx 3 $(3x+y-5)^2 = c(2x+y-3)$
Ex 4 $5x-10y+11\log(15x-5y-13) = c$
Ex 5 $(x+y)^2+6x-2y+2\log(x+y+1) = c$

Art 10 page 14 Ex. 4 $3x^4y^{10}-5x^2y^{-2} = c$ Ex 5 $xy^2-1 = cxy$
Ex 6 $x^2y^2+y = cx$ Ex 7 $3x^{7/3}y^{-4}+x^{8/3}y^{-3} = c$

Art 11 page 15 Ex 2 $5xy = x^5+c$ Ex 3 $y\sqrt{(1+x^2)} = x+c$.
Ex 4 $(y-1)e^{x^2/2} = c$.

Art 12 page 16 Ex 3 $y(c-x) = e^{x^2/2}$
Ex 4 $x^2\{\sqrt{(1+y^2)}+y\} = y+c$

Examples at end Page 17

1 $y-x = c(1+xy)$ 2 $\log x-(y+2x)x/(y+x)^2 = c$

3 $y = e^x(\log x+c)$ 4 $2y^2+2xy+x^2 = cx$

5 $x^{-1} = \cos\theta(\log\cos\theta+c)$ 6 $x^2y^3+x^2y^2+x^3-y^3 = c$

7 $2y(x+a)^3 = b(x+a)^4+c$. 8 $x-2y+4\log(2x-5y+10) = c$

9 $xy = ce^x(xy+1)$ 10 $(r^2+1)\sin^2\theta = c$ 11 $(y-1)\tan x = c-x$

12. $e^{-x/y}+y = c.$ 13. $\tan^2 s + t\cot s - \sin(s+t) - c^t = c.$

14. $2xy+1 = cy^2.$ 15. $x^3+2xy^3 = cy.$ 16. $8xe^{3y} = e^{8y}+c.$

17. $\sin(y/x) = cx.$ 18. $b(a^2x+b^2y) = a\tan(abx+c).$

19. $\sqrt{y}\log x = (\log x)^2+c.$ 20. $\log(x^2+y^2)-2/y = c.$

21. $y/x = \tan\{c-a/\sqrt{(x^2+y^2)}\}.$ 22. $y^3 = ce^{x^4/4}-1.$

23. $x^2+y^2+4xy+2x-8y = c.$ 24. $x\log y = xe^x-e^x+c.$

25. $y^2e^{-2/x} = 2\int x^2e^{-2/x}\,dx + c.$

26. $r^2(\sec\theta+\tan\theta) = \sec\theta+\tan\theta-\theta+c.$

27. $(1+x)(1-x+y) = ce^y.$ 28. $x^{3/2}y^2+x^2y^{4/3} = c.$

29. $\log(y+2)+2\tan^{-1}(y-2)\,(x-5) = c.$ 30. $e^y = (c-y^2)\tan x.$

CHAPTER III

Art. 14, page 20. Ex. 3. $(y-c)(y-2x-c)(y-4x-c) = 0.$

Ex. 4. $y = \sin(x+c).$ Ex. 5. $(2y-x^2-c)(y-ce^x) = 0.$

Ex. 6. $(y-cx)(y^2-x^2-c) = 0.$

Art. 15, page 21. Ex. 3. $2y = x^4+cx^2.$ Ex. 4. $xy-c^2x+c = 0.$

Art. 16, page 22. Ex. 2. $(y+1)^3(p^3-1)^2 = c$, with the given equation.

Ex. 3. $2x+y^3 = cy.$

Art. 17, page 23. Ex. 2. $y = cx+2c/(c^2+1).$

Ex. 3. $y^3 = 3cx+c^{3/2}.$ Ex. 4. $\log y = cx+c^2$

Examples at end. Page 23.

1. $(y-ce^{3x})(x^2+4y-c) = 0.$ 2. $c^2x^2+a^2 = 2yc.$

3. $y^2 = cx+c^2/4.$ 4. $e^{y/p} = cp$, with the given equation.

5. $e^y = ce^x+c^n.$ 6. $x = a\sec(y+c).$

7. $x+c = 2\tan^{-1}p-3/p$, with the given equation.

8. $4yc = (xc+4m)^2.$ 9. $(2y-x^2-c)(2y+x^2+c)(xy-cy-1) = 0.$

10. $y^2 = cx^2+c^2.$

11. $\log(2P-1)x-1/(2P-1) = c$ where $P^2 = (4y^2-x^2)/4x^2.$

12. $\{y^2+y\sqrt{(x^2+y^2)}\}/x^2+\log\{y+\sqrt{(x^2+y^2)}\}/cx^{-1} = 0.$

13. $y^3 = cx+c^2.$ 14. $4y+2x^2 = 4cx-c^2.$ 15. $2y = x(ce^x-c^{-1}e^{-x}).$

16. $(y-x-c)(y-ce^x)(1-x+y-ce^{-x}) = 0.$

17. $r = a\sin(\theta+c)$, in polar coordinates.

18. $x+p = ce^{-p}$, with the given equation.

19. $y^2 = cx^3-a^2c/(1+c).$ 20. $x^2+y^2+2\sqrt{2}cx+c^2 = 0.$

CHAPTER IV

Art. 19, page 27. Ex. 1. $x^2+4y = 0$, a singular solution.

Ex. 2. $y(y-x) = 0$; $y = x$, a singular solution; $y = 0$, not a solution.

Ex. 3. $y^2(x^2-y^2) = 0$; $x-y = 0$ and $x+y = 0$ are singular solutions; $y = 0$, not a solution.

Ex. 4. $9y^2 = 4x^3$, a singular solution.

Art 20 page 28 Ex 1 $x^2+y^2=1$

Ex 2 For ex 3, art 19 $x^2+y^2+2cx+c^2/2=0$,

For ex 4 art. 19, $3y-3cx+c^3=0$

Examples at end Page 33

1 $y=cx+a/c$ $y^2=4ax$ 2 $(y-cx)^2+a^2c=0$, $4xy=a^2$

3 $y-cx=\sqrt{(1-c^2)}-c\cos^{-1}c$, $y=\sin x$

4 $y+c=e^{x+c}$, $y=x+1$ 5 $27(y-cx)^2=4c^3$, $y=x^3$

6 $(3y+c)^2=2cx^3$, $6y=x^3$, $x=0$, cusp locus.

7 $(y+c)^2=(x+c)^3$, $x-y=4/27$, $x=y$, cusp locus

8 $x^2+y^2-2c(x+y)+c^2=0$ $x=0$ $y=0$, $x=y$, tac locus

9 $y-cx=ac/(c-1)$ $(x+y-a)^2=4xy$

10 $y^2=(x-c)^3$ $y=0$, cusp locus and envelope

11 $y=c(x-c)^2$ $27y=4x^3$

12 $x^2+y^2-4cx+5c^2-1=0$, $x^2/5+y^2=1$

13 $y^2+c^2x^2=c$, $4x^2y^2=1$

14 $cx^2+y^2=c/(c-1)$, $(x^2+y^2-1)^2-4x^2y^2=0$ (conics touching the four sides of a square).

15 $y-cx=ac/\sqrt{(1+c^2)}$ $x^{2/3}+y^{2/3}=a^{2/3}$

16 $9(8y+9/c)^2=256cx^3$, $3x+2y=0$

CHAPTER V

Art 23 page 36 Ex 4 $500\,000(1\cdot 2)^{t-1}$

Ex 5 $(\log 20)/\log 2=4\cdot 32$ hours

Art 24 page 37 Ex. 2 $50(1-e^{-3/8})$ lb

Ex 3 $x=\frac{15}{8}(2^{(t/10)+1}-2)/(2^{(t/10)+1}-1)$ Ex 4 74 minutes 46° C

Art 25 page 39 Ex 2 $y^2=2ax+c$ Ex 3 $y=ce^{x/a}$

Ex 4 $y^2-x^2=c$ Ex 5 $r=ce^{\theta/\sqrt{(k^2-1)}}$ Ex 6 $r=k\sin\theta$

Art 26, page 41 Ex 2 $y=cx^2$ Ex 3 $\theta^2+(\log r)^2=c$

Ex 4 $\log\sqrt{(x^2+y^2)}+\tan^{-1}y/x=c$ or $r=ke^{-\theta}$

Examples at end Page 44

1 $x^2+y^2=cy$ 2 $x^2+y^2=cx$ 3 $(x-y)^2-2a(x+y)+a^2=0$

4 $xy=A/2$ 5 $r=ce^{\sqrt{n}\,\theta}$ 6 $x^2+y^2=2\log cx$

7 $2x^2+y^2=c$ 8 $x^2-y^2=c$ 9 $(x-c)^2+y^2=c^2-1$

10 Circle centre midway between the given points, radius half the given sum

11 $x^2/(a^2+k^2)+y^2/k^2=1$ (the given points $(a,0)$, $(-a,0)$ the given rectangle $=k^2$)

12 $r=c(1-\cos\theta)$ 13 $r=ce^{\theta^2/2}$ 14 $r=c(1+\cos\theta)$

15 $x^{2/3}+y^{2/3}=a^{2/3}$ 18 $15\sqrt[3]{2}/(\sqrt[3]{2}-1)$ 19 4·98 per cent

20 45½ 21 $1\cdot e^{-25\pi/144}$ 22 $10/e$ 23 29° C

24 $1\cdot 09T=9+100(0\cdot 455)^t$ 25 $y=x\tan\alpha-x^2(g/2v^2)\sec^2\alpha$

26. $cx^2-y^2 = a^2c/(1+c)$ with fixed points $(-a, 0)$ and $(a, 0)$.

27. Portions of two parabolas.

CHAPTER VI

Art. 29, page 47. Ex. 2. $y = c_1 e^{x}+c_2 e^{3x}+c_3 e^{-4x}$.

Ex. 3. $y = c_1 e^{(1+\sqrt{2}/2)x}+c_2 e^{(1-\sqrt{2}/2)x}$.

Art. 31, page 49. Ex. 2. $y = e^{-3x/2}\left(A\cos\frac{\sqrt{7}}{2}x+B\sin\frac{\sqrt{7}}{2}x\right)$.

Ex. 3. $y = e^{5x}(A\cos 6x+B\sin 6x)$.

Art. 33, page 51. Ex. 2. $y = e^{ax}(c_1 x+c_2)+e^{bx}(c_3 x+c_4)$.

Art. 35, page 54. Ex. 2. $y = 5/2+c_1 e^{-x/3}+c_2 e^{-2x}$.

Ex. 3. $y = -x/4+c_1 e^{2x}+c_2 e^{-2x}$.

Art. 36, page 55. Ex. 2. $y = e^{3x}/7+c_1 e^{2x/3}+c_2 e^{-4x}$.

Ex. 3. $y = e^{bx}/(b^3-a^3)+c_1 e^{ax}+e^{-ax/2}\left(c_2\cos\frac{\sqrt{3}}{2}ax+c_3\sin\frac{\sqrt{3}}{2}ax\right)$.

Art. 37, page 57. Ex. 2. $y = e^{-2x}(c_1 x+c_2-\sin x)$.

Ex. 3. $y = xe^{x}/3+c_1 e^{x}+e^{-x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)$.

Ex. 4. $y = e^{2x}(x^2/2-x+1)+c_1 e^{x}+c_2 e^{2x}$.

Art. 38, page 59. Ex. 1. $y = \frac{16}{85}\sin\frac{1}{2}x+c_1 e^{x}+c_2 e^{-x}+c_3 e^{2x}+c_4 e^{-2x}$.

Ex. 2. $y = \sin x+e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)$.

Ex. 3. $y = x(\sin 3x-\cos 2x)+c_1\cos 2x+c_2\sin 2x+c_3\cos 3x+c_4\sin 3x$.

Art. 39, page 60. Ex. 2. $y = \frac{1}{8}(2x^5-10x^3+15x)+c_1\cos 2x+c_2\sin 2x$.

Ex. 3. $y = 2x^3+3x^2+27x+c_1+c_2 e^{x}+c_3 e^{2x}$.

Examples at end. Page 60.

1. $10+\cos\frac{\sqrt{3}}{2}x(c_1 e^{x/2}+c_2 e^{-x/2})+\sin\frac{\sqrt{3}}{2}x(c_3 e^{x/2}+c_4 e^{-x/2})$.
2. $\frac{1}{17}(8\sin 4x-15\cos 4x)+c_1 e^{-x}+c_2 xe^{-x}$.
3. $xe^{2x}+c_1 e^{2x}+c_2 e^{-7x/3}$. 4. $x+1+c_1 e^{-x}+c_2 e^{x}+c_3 xe^{x}$.
5. $e^{3x}(x^3-6x)+e^{3x}(A\cos x+B\sin x)$.
6. $x^4+8x^3+48x^2+168x+288+c_1 e^{x}+c_2 e^{\frac{-1+\sqrt{5}}{2}x}+c_3 e^{\frac{-1-\sqrt{5}}{2}x}$.
7. $\frac{1}{2}e^{-x}(\sin x+\cos x)+c_1 e^{-2x}+e^{-\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)$.
8. $e^{ax}(c_1+c_2 x+c_3 x^2+c_4 x^3+x^4/24)$.
9. $(x\cos 5x)/20+(x\cos x)/4+c_1\cos 5x+c_2\sin 5x+c_3\cos x+c_4\sin x$.
10. $e^{ax}x^{a+2}/(a+1)(a+2)+e^{ax}(c_1 x+c_2)$.
11. $\frac{1}{2}x^2e^{-2x}+c_1 e^{-3x}+e^{-2x}(c_2 x+c_3)$.
12. $x\cos 2x+c_1\cos x+c_2\sin x+c_3\cos 2x+c_4\sin 2x$.
13. $e^{2x/3}(x+c_1)-e^{-5x/2}(x+c_2)$. 14. $e^{x}(c_1+c_2 x-2\cos x-x\sin x)$.

15 $c_1e^{3x}+c_2e^{-3x}-(1+3x^2+3x^4)$

16 $\frac{1}{4}\sin x-\cos 2x+1/5+c_1\sin\sqrt{5}x+c_2\cos\sqrt{5}x$

18 $(x\cos x)/3+(2\sin x)/9+c_1\cos 2x+c_2\sin 2x$

19 $-x^2\cos x-x^2\sin x-2x\cos x+\sin x-\cos x+ce^x$

20 $x^2\sin x+x\cos x+c_1\sin x+c_2\cos x$

21 $-x^2\cos x+x\sin x+(c_1+c_2x)(c_3\cos x+c_4\sin x)$

22 $x^4-10x^3+48x^2-132x+c_1+c_2e^{-x}+c_3xe^{-x}$

23 $\frac{1}{4}e^{-2x}(3x\sin 2x+\cos 4x)+e^{-2x}(A\cos 2x+B\sin 2x)$

24 $c_1e^{-x}+c_2e^{-3x/2}-e^{-x}(\frac{1}{2}\sin\frac{1}{2}x+x^2+8)$

25 $x^4-9x^2+x\sin x+c_1+c_2x+c_3\cos x+c_4\sin x$

26 $1/8+(\cos 2x)/3-(\cos 4x)/6+c_1\cos 2\sqrt{3}x+c_2\sin 2\sqrt{3}x$

CHAPTER VII

Art 40 page 63 Ex 1 $y=c_1x+c_2x^4$ Ex 2 $y=x^{-4}(c_1+c_2\log x)$

Ex 3 $y=x(A\cos\log x+B\sin\log x)$

Art 41, page 64 Ex 2 $y=e^x(x^{-2}-2x^{-3})+c_1x^{-2}+c_2x^{-3}$

Ex 3 $y=x^2/30+3x/10-2+x^{-1}(c_1+c_2\cos\log x+c_3\sin\log x)$

Art 42 page 67 Ex 3 $(x^4-4x^2)y=c_1x^3+c_2x+c_3$

Ex. 4 $y=c_1x+c_2xe^{-1/2x^2}$

Art 44 page 73 Ex 5

$$y=a\left(1+\frac{2}{1\cdot 1}x+\frac{2^2}{2!\,1\cdot 4}x^2+\frac{2^3}{3!\,1\cdot 4\cdot 7}x^3+\quad\right)+$$
$$+bx^{2/3}\left(1+\frac{2}{1\cdot 5}x+\frac{2^2}{2!\,5\cdot 8}x^2+\frac{2^3}{3!\,5\cdot 8\cdot 11}x^3+\quad\right)$$

Ex 6

$$y=a(1\ 2+2\ 3x+3\ 4x^2+4\ 5x^3+\quad)+$$
$$+b\sqrt{x}(3\ 5+5\ 7x+7\ 9x^2+9\ 11x^3+\quad)$$

Ex 7

$$y=a(3x-5x^3)+$$
$$+b\left(1+\frac{-3\cdot 4}{2!}x^2+\frac{-3\cdot -1\cdot 4\cdot 6}{4!}x^4+\frac{-3\cdot -1\cdot 1\cdot 4\cdot 6\cdot 8}{6!}x^6+\quad\right)$$

Examples at end. Page 74

1 $7/5+c_1x+c_2x^6$ 2 $-(x+1)^4/6+c_1(x+1)^5+c_2(x+1)^{-2}$

3 $(x^4-4x^2)y=x^4+c_1x+c_2$

4 $(x^2\log x)/12+c_1x^2+x^{-1}\{c_2\cos(\sqrt{3}\log x)+c_3\sin(\sqrt{3}\log x)\}$

5 $-x^4/2+c_1+c_2x^2+c_3x^4$ 6 $a(x+3x^2/4)$

7 $x^n\left(1-\frac{x^2}{2^2(n+1)}+\frac{x^4}{2^4 2\,(n+1)(n+2)}-\quad\right)$

8 $ax+b(1-x^2-x^4/3-x^6/5-\quad)$

9 $(ax/4+7b/12)/a^2+c_1\sqrt{(ax+b)}+c_2(ax+b)^{-2}$

10 $x^2y^2-xy^2=cx+c_1$ 11 $15y=\sqrt{x}\log x+c_1\sqrt{x}+c_2x^{-7}$

CHAPTER VIII

Art. 46, page 76. Ex. 2. $y = x/c - \dfrac{c^2+1}{c^2}\log(1+cx)+c_1$.

Ex. 3. $y = -\sin(x+c)+c_1x+c_2$. Ex. 4. $y = x^2+c\log x+c_1$.

Art. 47, page 77. Ex. 1. $y-c\log(y+c) = x+c_1$.

Ex. 2. $cy-1 = c^2(x+c_1)^2/4$. Ex. 3. $\log(cy+c^2) = c(x+c_1)$.

Art. 48, page 77. Ex. 1. $y = c\sin(ax+c_1)$. Ex. 2. $2y^{-1} = (ax+c)^2$.

Ex. 3. $2e^{-y/2} = c-x$. Ex. 4. $y = \sec(x+c)$.

Art. 49, page 79. Ex. 2. $y = -x^3/4+c_1xe^{x^2}+c_2x$.

Ex. 3. $y = x^3(c_1+c_2e^x)$. Ex. 4. $y = -e^{2x}/2+c_1e^x+c_2e^x\int e^{x+x^2}\,dx$.

Art. 50, page 80. Ex. 2. $y = xe^{x/2}(c_1+c_2x)$.

Ex. 3. $y = (c_1x^4+c_2x^{-3})/(x-2)$.

Art. 51, page 81. Ex. 2. $y = (e^x+e^{2x})\log(1+e^{-x})+c_1e^x+c_2e^{2x}$.

Ex. 3. $y = x^3\tan^{-1}x-\frac{1}{2}x^2\log(1+x^2)+c_1x^2+c_2x^3$.

Art. 52, page 82. Ex. 2. $y = (\sin^3x)/4+c_1\sin^2x+c_2\csc x$.

Ex. 3. $y = -x^2/a^2-12x/a^4-24/a^6+c_1e^{a\sqrt{x}}+c_2e^{-a\sqrt{x}}$.

Examples at end. Page 82.

1. $y+c_1 = \log\sec(x+c)$. 2. $y = e^x\{c(x+1)^5-x/4+c_1\}$.
3. $\tan y/2 = ce^{\sqrt{2x}}$. 4. $y = -x^2+cx\int e^{-1/x}\,dx+c_1x$.
5. $y = cx^2\int x^{-4}e^{-x^2/2}\,dx+c_1x^2$. 6. $y = (x+a)^6/12+c(x+a)^4+c_1$.
7. $y = cx^{-1}\int e^{-x^2/2}x^2\,dx+c_1x^{-1}$.
8. $y = \sin x\int e^x(\csc x+c\csc^2x)\,dx+c_1\sin x$.
9. $y = e^{3x^2/2}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)$. 10. $x+c_1 = \log(e^{1/y}-c)$.
11. (i) $y = -\sin x^2+c_1e^{x^2}+c_2e^{-x^2}$, (ii) $y = \{c_1(1-x^2)+c_2x\}/(1+x^2)$.
12. $y = e^x+c_1e^x/x+c_2/x$.
13. (i) $y = c_1(2x+7)+c_2e^{2x}-\frac{1}{2}+e^{2x}\int e^{-2x}(x+3)\log(x+3)\,dx$.
    (ii) $y = c_1e^x+e^x\int e^{-x}(x+1)^2(x+c_2)\,dx$.

CHAPTER IX

Art. 53, page 85. Ex. 2. $(x-a)^2+(y-b)^2 = r^2$.

Art. 54, page 87. Ex. 2. A parabola.

Art. 55, page 89. Ex. 3. $s = (g/k^2)(kt-1+e^{-kt})$.

Ex. 4. 2 seconds nearly; 19·5 feet per second.

Examples at end. Page 92.

1. $c\sin^{-1}\sqrt{(y/c)}-\sqrt{\{y(c-y)\}} = x+c_1$. 2. $2y = c_1e^{cx}+c_1^{-1}e^{-cx}$.
3. $y+c_1 = \log\dfrac{1+\sqrt{\{1-(c-x)^2\}}}{c-x}-\sqrt{\{1-(c-x)^2\}}$.
4. $y = e^{x/a}+e^{-x/a}$.

5 $2(y+k) = k(e^{x/k}+e^{-x/k})$ if curve touches y-axis at origin

6 $y = kx^3/6T_0$ 7 $y = \frac{1}{2}a(e^{\sqrt{(w/T)}x}+e^{-\sqrt{(w/T)}x})$

8 $s = Ae^{-t/2}\cos\left(\frac{\sqrt{17}}{2}t+\gamma\right)+\frac{8}{17}(4\sin 2t+\cos 2t)$, $4\pi/\sqrt{17}$, π, $16/\sqrt{17}$.

9 $t = \int_7^6 dx\Big/\sqrt{\left\{2g\left(x-7-16\log\frac{x+8}{15}\right)\right\}}$

10 $v = \sqrt{(g/k)}(e^{2\sqrt{(gk)}t}-1)/(e^{2\sqrt{(gk)}t}+1)$

$$s-s_0 = \frac{1}{2k}\log\tfrac{1}{4}(1+e^{2\sqrt{(gk)}t})(1+e^{-2\sqrt{(gk)}t})$$

11 $t = \frac{1}{4}\pi\sqrt{\frac{R}{g}}$, $v = \sqrt{(gR)}$ 12 $a^2(a^2-s^2) = kt^2$

CHAPTER X

Art. 58, page 96 Ex 3 $y = cx$, $x^2 = c_1x$

Ex 4 $(x+a)^2-(y+b)^2 = k_1$, $x+y+z+a+b+c = k_2(z+c)$

Ex 5 $x^2-z^2 = c_1$, $x^2+z^2 = c_2y^2$

Art 59, page 98 Ex 2 $y = c_1e^x+c_2e^{-x}$, $z = c_1e^x-c_2e^{-x}$

Ex 3 $y = 3\cos x+c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x}$,

$z = 2\sin x+c_1(1+\sqrt{2})e^{\sqrt{2}x}-c_2(\sqrt{2}-1)e^{-\sqrt{2}x}$

Ex 4 $x = 3t-2+19c_1e^{-11t/6}$, $y = 5t+3+6c_1e^{-11t/6}+c_2e^t$

Art 62, page 102 Ex 3 $(x+z)y = c(y+z)$

Ex 4 $x^2y-3xz+2yz^2 = c$ Ex. 5 $x/y+y/z = c$

Ex 6 $x^2-xy+y^2 = cz$.

Art 63, page 103 Ex 2 The intersections with the hyperboloid of the cylinders $y^2 = 4a(x-k)$.

Ex. 3 The intersections with the given surface of the quadrics $xy-z = c$

Examples at end. Page 103.

1 $x^2-y^2 = c_1z$, $y^2-z^2 = c_2x$

2 $x+y+z = a$, $x^2+y^2+z^2 = b$

3 $x^2-y^2 = a$, $(x-y)z^{1/a} = b$

4 $x^2y = a$, $\sec(a+z)+\tan(a+z) = bx$ 5 $y = ax$, $az-2 = be^x$

6. $y = 1/12-x/2+11e^x/12-3c_1e^{-3x}/2-c_2e^{-2x}$,

$z = 1/36+x/6-e^x/6+c_1e^{-3x}+c_2e^{-2x}$

7 $y = 2x\cos x-(3x\sin x)/2+c_1\cos x+c_2\sin x$,

$z = 4x\sin x-(9x\cos x)/2+c_3\cos x+c_4\sin x$, where

$5c_3 = -12c_1-c_2+1$, $5c_4 = c_1-12c_2-1/2$

8 $x = e^{-t}\cos t+c_1e^{-2t}+c_2$ $y = e^{-t}\sin t+c_1e^{-2t}+c_3$

9 $(y-z)(z-x) = c$ 10 $(x-c)^2+y^2+z^2 = 1$

11 $(x^2+y^2+z^2-c)(x^2+y^2-z^2-c) = 0$ 12 $x/y+y/z+z/x = c$.

13. $x^2+y^2+z^2 = cz$. 14. $xy/(x+z)(y+z) = a/(a+z^2)$.
15. $(x^2-y)/(y^2-z) = a$. 16. $(x+z)/y+(y+z)/x = a$.
17. $x^2+y^2+z^2 = 3z,\ y = z$.
18. The intersections of the ellipsoid with the spheres $x^2+y^2+z^2 = c$.
19. The intersections of the cylinders with the parabolic cylinders $y^2 = ax$.
21. $xy+xz+xt+yz+yt+zt = c$.
22. $xy^2+3x^2zt-2y^3z^2-yt^3+4zt^2 = c$.

CHAPTER XI

Art. 64, page 106. Ex. 2. $(x+zp)(1+q) = (y+zq)(1+p)$.
Ex. 3. $z = px+qy+pq$. Ex. 4. $z = pq$.
Art. 65, page 107. Ex. 1. $px = qy$. Ex. 2. $bp = aq$. Ex. 3. $yp = xq$.
Ex. 4. $xp+yq = z$.
Art. 66, page 110. Ex. 3. $\Phi\left(\dfrac{y-b}{x-a}, \dfrac{z-c}{x-a}\right) = 0$. Ex. 4. $\Phi(y/x, z) = 0$.
Art. 67, page 111. Ex. 1. $z = ax+y\sqrt{(4-a^2)}+c$.
Ex. 2. $\log z-a/z = x+ay+b$. Ex. 3. $z = ax+by+4a-b$.
Ex. 4. $z = a(x^2+y^2)+b$. Ex. 5. $z = ax+ay/(a-1)+c$.
Ex. 6. $x+a^2y+b = a\log z$. Ex. 7. $z+a\sqrt{(1-x^2)}-a\tan^{-1}y = b$.
Art. 68, page 114. Ex. 3. A complete integral, $a^2z = (x+ay+b)^2$.
Examples at end. Page 116.

1. $F\{(x+y+z)(y-x)^2, (y-x)/(z-y)\} = 0$.
2. $F(x^2+y^2+2z, xy) = 0$. 3. $F(x_1/a-x_2/b, x_1/a-x_3/c, x_1/a-z) = 0$.
4. $(z-bx-by-c)(z-b^2x-by-c) = 0$.
5. $z = a(x+1)^2/2+(y-1)^2/2a+b$.
6. $z = \sin\{(x+ay)/(1-a)+b\}$. 7. $z = a\log x\sec y+b$.
8. $(z-b)\{z^2+a^2-(x+ay+b)^2\} = 0$.
9. $z = a\tan^{-1}x+\tfrac{1}{2}a\log\dfrac{1+y}{1-y}+b$. 10. $F\{(x^2+y^2+z^2)/z, y/z\} = 0$.
11. $F\{\sqrt{(x^2+y^2)}-z, y/x\} = 0$.
12. $F\{(x_1-x_2)/(x_2-x_3), x_1+x_2+x_3, z\} = 0$.
13. $z^2 = ax+by+a^2+b$. 14. $F(x^2-y-z, xy-z^2) = 0$.
15. $F\{(x-y)/(y-z), (y-z)/(z-x)\} = 0$.
16. $2z+b = x^2+y^2+x\sqrt{(x^2+a-1)}+(a-1)\log\{x+\sqrt{(x^2+a-1)}\}+$
$+y\sqrt{(y^2-a)}-a\log\{y+\sqrt{(y^2-a)}\}$.
17. $2y^2z+y^2(a-x)^2+y^4 = b$.
19. $F\{(y-y_0)/(x-x_0), (z-z_0)/(x-x_0)\} = 0$. (All cones with vertex (x_0, y_0, z_0).)
20. $xyz = 2k/9$, where k is the volume.
21. $(ax+by-z)(a+b-ab) = abk$, where k is the given sum.

CHAPTER XII

Art 70, page 118 Ex 2 $s = 0$ Ex. 3 $xy^2r - x^2yt = y^2p - x^2q$

Ex 4. $rq - sp = 0$ Ex. 5. $z = x^2y + f(x) + \phi(y)$.

Ex. 6 $z = f(y)\cos ax + \phi(y)\sin ax$

Art 71, page 119 Ex 2. $z = Be^{-k^2y}\sin(kx/a)$.

Ex. 3 $y = B\sin px \sin apt$ Ex 4. $z = e^{(y-ax)(1-a)}$

Art. 72, page 121 Ex. 1 $z = F_1(y) + F_2(y+x)$

Ex. 2 $z = xF_1(y-3x) + F_2(y-3x)$

Ex 3 $z = F_1(y) + F_2(y-x) + F_3(y+3x)$

Ex 4 $z = F(y+x) + \phi(y+ix) + x\psi(y+ix) + \phi(y-ix) - x\psi(y-ix)$.

Art 73 page 122. Ex 2. $z = x^4y/12 - x^5/15 + xF_1(y-2x) + F_2(y-2x)$

Ex. 3 $z = (\sin y)\ 36 + (x\cos y)\ 6 + F_1(y+2x) + F_2(y-3x)$

Ex 4 $z = (e^x\sin y)/2 + F_1(y+x) + F_2(y-x)$

Page 124 Ex. 7 $z = (5x+y)^3/540 + F_1(y) + F_2(y+3x) + F_3(y-4x)$

Ex. 8 $z = \{2(3x-4y)^3\log(3x-4y) - 3(3x-4y)^3\}/2116 +$
$+ xF_1(y+5x) + F_2(y+5x)$

Examples at end. Page 133

1 $z = e^{my}$ or $z = e^{lx+(1-l)y}$ 2 $z = e^{lx+my}$ where $m = l(1+l)(1-l)$

3 $z = e^{l(x+y)}$ or $z = e^{lx-(l-1)y}$

4 $z = x^3/6 - x^2y/2 + F_1(y+x) + F_2(y-x)$

5 $z = 23x^3/216 - x^2y/12 + F_1(2y+x) + F_2(3y-7x)$

6 $z = x^3y/6 + x^4/24 + F_1(y+3x) + F_2(y-2x)$

7 $z = \frac{1}{4}\cos(x-y) + F_1(y+x) + xF_2(y+x) + x^2F_3(y+x)$

9 $V = XYZ$,

$$\text{where}\quad X = f(x,k) = \begin{cases} c_1e^{\sqrt{k}x} + c_2e^{-\sqrt{k}x} & \text{if } k > 0, \\ c_1x + c_2 & \text{if } k = 0, \\ c_1\cos\sqrt{(-k)}x + c_2\sin\sqrt{(-k)}x & \text{if } k < 0, \end{cases}$$

$Y = f(y, l)$, $Z = f(z, m)$ and $k + l + m = 0$

10 $x = 2\{\sin x - (\sin 2x)/2 + (\sin 3x)/3 - \ldots\}$

11 $x = \frac{\pi}{2} - \frac{4}{\pi}\{(\cos x)/1^2 + (\cos 3x)/3^2 + (\cos 5x)/5^2 + \ldots\}$

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